

On higher spin partition functions

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ABSTRACT: We observe that the partition function of the set of all free massless higher spins $s = 0, 1, 2, 3, \dots$ in flat space is equal to one: the ghost determinants cancel against the "physical" ones or, equivalently, the (regularized) total number of degrees of freedom vanishes. This reflects large underlying gauge symmetry and suggests analogy with supersymmetric or topological theory. The $Z = 1$ property extends also to the AdS background, i.e. the 1-loop vacuum partition function of Vasiliev theory is equal to 1 (assuming a particular regularization of the sum over spins); this was noticed earlier as a consistency requirement for the vectorial AdS/CFT duality. We find that $Z = 1$ is true also in the conformal higher spin theory (with higher-derivative ∂^{2s} kinetic terms) expanded near flat or conformally flat S^4 background. We also consider the partition function of free conformal theory of symmetric traceless rank s tensor field which has 2-derivative kinetic term but only scalar gauge invariance in flat 4d space. This non-unitary theory has Weyl-invariant action in curved background and it corresponds to "partially massless" field in AdS_5 . We discuss in detail the special case of $s = 2$ (or "conformal graviton"), compute the corresponding conformal anomaly coefficients and compare them with previously found expressions for generic representations of conformal group in 4 dimensions.

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1 Introduction

Higher spin theories containing infinite number of fields [1, 2] may have novel unexpected properties at the quantum level related to the fact that one is required to sum an infinite number of individual field contributions. This summation requires a particular regularization prescription that should be consistent with underlying symmetries of the theory. Some examples were discussed in [3, 4] and especially in [5–11] which we will elaborate on.

Our aim will be to study free partition functions in flat and conformally flat backgrounds for infinite families of higher spin fields. In addition to the Fronsdal massless higher spin (MHS) fields with standard 2-derivative kinetic terms we will consider higher derivative conformal higher spin (CHS) fields [12] and also conformal symmetric tensor (CST) fields with 2-derivative Weyl-invariant actions [13].

We shall start in section 2.1 with a simple but remarkable observation that the flat-space partition function of the free MHS theory containing each $s = 0, 1, 2, \dots$ spin once is trivial, or, equivalently, its regularized total number of dynamical degrees of freedom vanishes. This reflects the presence of a large gauge symmetry, with the contribution of the determinant of spin s kinetic operator cancelling against that of the ghost determinant for the spin $s + 1$ field. This one-loop $Z = 1$ property generalizes to the AdS_d vacuum background of Vasiliev theory provided one uses a special regularization prescription [6, 8]. In particular, we will find that for even dimension d there are special mass sum rules implying the cancellation of not only logarithmic [6, 8] but also power UV divergences.

In section 2.2 we will show that the $Z = 1$ property holds also in the CHS theory on flat background: here the determinants do not cancel automatically but the regularized number of dynamical degrees of freedom still vanishes. As was found in [5, 7, 8], the regularized sums over s of the conformal anomaly a_s and c_s coefficients of the CHS theory vanish, implying its one-loop UV finiteness in 4 dimensions and suggesting that its total one-loop partition function on a conformally-flat background should be trivial. We shall consider the case of the CHS theory on S^4 where for each spin s its partition function should be given by the ratio (2.30) of MHS partition functions with alternative boundary conditions [5, 7, 11]. This relation will be verified in Appendix A following the dimensional regularization approach used in the spin 0 case in [14]. Summing over all spins implies then that $[Z_{\text{MHS}}(AdS_5)]_{\text{tot}} = 1$ is directly related to $[Z_{\text{CHS}}(S^4)]_{\text{tot}} = 1$.

In section 3 we shall study 2-derivative Weyl-invariant "higher spin" actions for symmetric traceless rank s fields in 4 dimensions that have only scalar gauge invariance in conformally-flat background. This CST theory is non-unitary (though for a different reason than CHS one) and may be viewed as a "maximal depth" $r = s$ representative of a family of conformal higher spin fields with rank $s - r$ tensor gauge invariance [15, 16] (with CHS case being "minimal depth" case $r = 1$). In flat $d = 4$ space the number of dynamical degrees of freedom of rank s CST field happens to be the same $s(s + 1)$ as of a CHS spin s field and thus the total regularized Z_{CST} is again equal to 1. The associated $SO(2, 4)$ conformal group representation is $(3; \frac{s}{2}, \frac{s}{2})$ that corresponds to "maximal-depth" partially massless [16] spin s field in AdS_5 . We will consider the CST fields defined on a

curved 4d background and compute the corresponding partition function and conformal anomaly a_s and c_s coefficients comparing them with the general expressions for representation $(\Delta; \frac{s}{2}, \frac{s}{2})$ field given in [11]. Some details about scalar gauge invariance in curved space and CST partition functions on $S^1 \times S^3$ and S^4 will be presented in Appendices B,C and D.

2 Summing over spins

2.1 Massless higher spins

2.1.1 Flat space

Let us consider the standard 2-derivative free massless higher spin (MHS) field in flat d -dimensional space. The corresponding partition function can be written as

$$Z_{\text{MHS},s} = \left[\frac{\det \Delta_{s-1\perp}}{\det \Delta_{s\perp}} \right]^{1/2} = \left[\frac{(\det \Delta_{s-1})^2}{\det \Delta_s \det \Delta_{s-2}} \right]^{1/2}, \quad (2.1)$$

where Δ_s is flat Laplacian $-\partial^2$ defined on symmetric rank s traceless tensors, and $\Delta_{s\perp}$ is its restriction to transverse fields. We shall assume that $\det \Delta_k$ with $k < 0$ is replaced by 1, i.e. $Z_{\text{MHS},0} = (\det \Delta_0)^{-1/2}$. Let us consider a theory where each massless field with $s = 0, 1, 2, \dots$ appears just once. This is field content of Vasiliev theory linearized near AdS_d vacuum which is dual to a large N free complex scalar theory in $d = d - 1$ dimensions. While the expansion of interacting Vasiliev theory near flat space is singular, we may formally view the free MHS partition function in flat space as a formal zero curvature limit of its one-loop counterpart in AdS_d . Then one finds that the total partition function is trivial:

$$(Z_{\text{MHS}})_{\text{tot}} = \prod_{s=0}^{\infty} Z_{\text{MHS},s} = \left[\frac{1}{\det \Delta_0} \right]^{1/2} \left[\frac{\det \Delta_0}{\det \Delta_{1\perp}} \right]^{1/2} \left[\frac{\det \Delta_{1\perp}}{\det \Delta_{2\perp}} \right]^{1/2} \left[\frac{\det \Delta_{2\perp}}{\det \Delta_{3\perp}} \right]^{1/2} \dots = 1. \quad (2.2)$$

This remarkable property reminds of a supersymmetric theory where the bosonic contribution to the vacuum partition function is cancelled against the fermionic one (implying the vanishing of the vacuum energy). Here the cancellation is between the contribution of the physical spin s field determinant and the ghost determinant for spin $s + 1$ field, i.e. it reflects a large gauge symmetry of the theory.¹

The cancellation of an infinite number of factors in (2.2) is formal (cf. $1-1+1-1+\dots=0$) as it depends on how one groups terms together: in general, an infinite product requires a regularization and depends on its choice. The choice of regularization should be consistent with an underlying symmetry of the theory (in the present case – higher spin gauge symmetry). Let us first consider the case of $d = 4$. Observing that each spin $s > 0$ field has 2 dynamical degrees of freedom (cf. (2.1))

$$Z_{\text{MHS},s} = (Z_0)^{\nu_s}, \quad Z_0 = \left[\frac{1}{\det \Delta_0} \right]^{1/2}, \quad \nu_s = (s+1)^2 + (s-1)^2 - 2s^2 = 2, \quad (2.3)$$

¹This also suggests an analogy with a topological theory. Similar examples are an antisymmetric tensor potential of rank d in $d + 1$ dimensions, Chern-Simons theory and 3d gravity.

we get

$$Z_{\text{tot}} = (Z_0)^{\nu_{\text{tot}}}, \quad \nu_{\text{tot}} = 1 + \sum_{s=1}^{\infty} \nu_s = 1 + 2 \sum_{s=1}^{\infty} 1 = 0, \quad (2.4)$$

where we used the standard Riemann zeta-function as a regularization of the sum over s : $\sum_{s=1}^{\infty} 1 = \zeta_R(0) = -\frac{1}{2}$. Remarkably, in $d = 4$ the use of the simple zeta-function regularization is thus equivalent to the formal cancellation of factors in (2.2).² Note that this flat-space property $(Z_{\text{MHS}})_{\text{tot}} = 1$ is non-trivial, i.e. it holds for a flat torus, implying, in particular, the vanishing of vacuum energy or finite temperature partition function on $S^1 \times R^3$. The partition function may be non-trivial if one considers an orbifold of flat space.

For a massless spin s field in d flat dimensions we get

$$\det \Delta_s = (\det \Delta_0)^{N_s}, \quad \det \Delta_{\perp s} = (\det \Delta_0)^{N_s^{\perp}}, \quad (2.5)$$

$$N_s = \binom{s+d-1}{s} - \binom{s+d-3}{s-2}, \quad N_s^{\perp} = N_s - N_{s-1}, \quad \nu_s = N_s^{\perp} - N_{s-1}^{\perp} = 2 \left[s + \frac{1}{2}(d-4) \right] \frac{(s+d-5)!}{s!(d-4)!}$$

Then $Z_{\text{tot}} = 1$ or $\nu_{\text{tot}} = 0$ is true, e.g., for general even d if one uses the following regularization:

$$\nu_{\text{tot}} = 1 + \sum_{s=1}^{\infty} \nu_s e^{-\epsilon \left[s + \frac{1}{2}(d-4) \right]} \Big|_{\text{fin.}} = 0. \quad (2.6)$$

Here one performs the sum for fixed ϵ , then takes $\epsilon \rightarrow 0$ and finally drops all singular $\frac{1}{\epsilon^n}$ terms (in $d = 4$ this is equivalent to the standard zeta-function prescription).³

2.1.2 Ricci-flat space

One may wonder if the $(Z_{\text{MHS}})_{\text{tot}} = 1$ property may generalize to curved spaces, e.g., Ricci-flat ones. As is well known, massless higher spin theories are not consistent in $R_{\mu\nu} = 0$ background (do not have flat-space gauge symmetries surviving) for $s > 2$. However, one may formally assume that there exists a consistent theory of all higher spin fields where proper gauge symmetry is present off-shell at interacting level. Vasiliev theory does not have $R_{\mu\nu} = 0$ as a classical vacuum solution since the limit of vanishing cosmological constant appears to be singular in the interaction terms, but one may consider formally expanding the (hopefully existing) action of Vasiliev theory near an off-shell $R_{\mu\nu} = 0$ background and computing the resulting one-loop partition function. If $R_{\mu\nu} = 0$ is not a classical solution this partition function will be gauge-dependent, but otherwise it may be well-defined and of interest being a direct generalization of the flat-space one (2.1),(2.2). A

²This is also reminiscent of the use of the zeta-function regularization in computing vacuum energy in bosonic string theory, where the use of $\zeta_R(-1) = -\frac{1}{12}$ leads to the value of the tachyon mass ensuring that the vector particle appearing on the first excited level is massless in $D = 26$, in agreement with symmetries of the critical string theory.

³ An alternative regularization that gives vanishing result in any d is to introduce a cutoff function $f(s, \epsilon)$ (with $f(s, 0) = 1$) for each $\Delta_{\perp s}$ factor in (2.1) separately thus getting

$$\nu_{\text{tot}} = 1 + \sum_{s=1}^{\infty} \left[f(s, \epsilon) N_s^{\perp} - f(s-1, \epsilon) N_{s-1}^{\perp} \right] = 0.$$

This prescription is the direct analog of the cancellation of the determinant factors in (2.2).

natural spin s counterpart of spin 2 Lichnerowicz operator on Ricci flat background may be chosen as in [17]⁴

$$\Delta_{Ls} = -\nabla_s^2 + X_s, \quad (X_s \varphi)^{\mu_1 \dots \mu_s} = -s(s-1) R_{\nu}^{(\mu_1} \lambda^{\mu_2} \varphi^{\mu_3 \dots \mu_s) \nu \lambda}. \quad (2.7)$$

Then one may formally consider the following generalization of the well-known spin $s = 1, 2$ partition functions on Ricci-flat background to any spin s [7]

$$Z_{\text{MHS},s} = \left[\frac{(\det \Delta_{Ls-1})^2}{\det \Delta_{Ls} \det \Delta_{Ls-2}} \right]^{1/2}. \quad (2.8)$$

Taking the product of (2.8) over s we again get that $(Z_{\text{MHS}})_{\text{tot}} = 1$ as in the flat space case (2.2):

$$\begin{aligned} (Z_{\text{MHS}})_{\text{tot}} &= \prod_{s=0}^{\infty} Z_{\text{MHS},s} \\ &= \left[\frac{1}{\det \Delta_{L0}} \right]^{1/2} \left[\frac{(\det \Delta_{L0})^2}{\det \Delta_{L1}} \right]^{1/2} \left[\frac{(\det \Delta_{L1})^2}{\det \Delta_{L2} \det \Delta_{L0}} \right]^{1/2} \left[\frac{(\det \Delta_{L2})^2}{\det \Delta_{L3} \det \Delta_{L1}} \right]^{1/2} \dots = 1. \end{aligned} \quad (2.9)$$

Here the cancellation follows from the fact that each spin s operator appears exactly twice in both the numerator and the denominator.

2.1.3 Conformally-flat case: AdS_d

The $Z_{\text{tot}} = 1$ property is expected to hold also in the proper vacuum of the Vasiliev theory – AdS_d space, and should be true to all orders in the coupling expansion. As was pointed out in [6, 8], this is the requirement of consistency of the vectorial AdS/CFT: the boundary theory of free $U(N)$ scalar has log of its partition function scaling as N which should match the classical action of the Vasiliev theory in AdS_d , while the 1-loop (and all higher-loop) corrections to $\ln Z_{\text{tot}}$ of MHS theory should vanish (in a proper regularization). To demonstrate this even at the one-loop order (free MHS theory in AdS_d background) is, however, much less trivial than in the flat space background considered above. Let us introduce the operator ($k = 0, 1, \dots, s-1$)

$$\Delta_s(M_{s,k}^2) \equiv -\nabla_s^2 + M_{s,k}^2 \sigma, \quad M_{s,k}^2 = s - (k-1)(k+d-2), \quad (2.10)$$

where $\sigma = \pm a^{-2} = \pm 1$ for unit-radius S^d or euclidean AdS_d space ($\sigma = 0$ in flat space). Let us then define the partition function of a "partially-massless" [20] spin s field (with gauge invariance with rank k tensor parameter) [7]

$$Z_{s,k} = \left[\frac{\det \Delta_{k\perp}(M_{k,s}^2)}{\det \Delta_{s\perp}(M_{s,k}^2)} \right]^{1/2}. \quad (2.11)$$

⁴This is of course a strong assumption (motivated just by simplicity) as a generalization of the Lichnerowicz operator coming out of a consistent higher spin theory formally expanded near a Ricci-flat background may contain also higher derivative terms with higher powers of the curvature tensor, cf. [18, 19].

Then for the *massless* spin s field (having maximal gauge invariance with rank $k = s - 1$ parameter) on a homogeneous conformally flat space we get the following counterpart of (2.1) (see [21, 4, 7, 22])

$$\begin{aligned} Z_{\text{MHS},s} \equiv Z_{s,s-1} &= \left[\frac{\det \Delta_{s-1 \perp} (M_{s-1,s}^2)}{\det \Delta_{s \perp} (M_{s,s-1}^2)} \right]^{1/2} \\ &= \left[\frac{(\det \Delta_{s-1} (M_{s-1,s}^2))^2}{\det \Delta_s (M_{s,s-1}^2) \det \Delta_{s-2} (M_{s+2,s+1}^2)} \right]^{1/2}, \end{aligned} \quad (2.12)$$

where $s > 0$ and for $s = 0$ we have $Z_{\text{MHS},0} = [\det(-\nabla^2 + M_0^2)]^{-1/2}$, $M_0^2 = M_{0,-1}^2 \sigma = 2(d-3)\sigma$.

Taking the product of (2.12) over s we conclude that there is no straightforward cancellations of determinant factors that happened in flat space in (2.2) or in (2.9): the same-spin operators that appear in the numerator and denominator of $\prod_{s=0}^{\infty} Z_{\text{MHS},s}$ are different for non-zero curvature, i.e. $\sigma \neq 0$: they have different mass-like terms in (2.10). For example, in $d = 4$ case we get

$$(Z_{\text{MHS}})_{\text{tot}} = \left[\frac{1}{\det \Delta_0(2)} \right]^{1/2} \left[\frac{\det \Delta_0(0)}{\det \Delta_{1 \perp}(3)} \right]^{1/2} \left[\frac{\det \Delta_{1 \perp}(-3)}{\det \Delta_{2 \perp}(2)} \right]^{1/2} \left[\frac{\det \Delta_{2 \perp}(-8)}{\det \Delta_{3 \perp}(-1)} \right]^{1/2} \dots \quad (2.13)$$

Using spectral zeta-function regularization one has, in a homogeneous space background,

$$\ln \det \Delta_s = -\zeta_{\Delta_s}(0) \ln(La)^2 - \zeta'_{\Delta_s}(0), \quad \zeta_{\Delta_s}(z) = \bar{\zeta}_{\Delta_s}(z) V_d. \quad (2.14)$$

Here L is UV cutoff, a is the curvature radius and V_d is the volume. In a non-compact space like AdS_d where the volume is formally divergent this relation requires an IR regularization (see, e.g., [14] and refs. there). Dropping power IR divergences, regularized V_d is then finite for even d and log divergent for odd d (R is an IR cutoff)

$$V_{d=\text{even}} = k_1, \quad V_{d=\text{odd}} = k_2 \ln R + k_3. \quad (2.15)$$

As was shown in [8] using the explicit form of higher spin heat kernel for AdS_d [23], keeping the argument z of spectral ζ -function non-zero, summing over spins and then taking $z \rightarrow 0$ one finds that $\bar{\zeta}_{\text{tot}}(z) = O(z^2)$, i.e. that $\bar{\zeta}_{\text{tot}}(0) = 0$, $\bar{\zeta}'_{\text{tot}}(0) = 0$. Thus for any AdS_d with $d > 3$ one has⁵

$$(Z_{\text{MHS}})_{\text{tot}} = \prod_{s=0}^{\infty} Z_{\text{MHS},s} = 1. \quad (2.16)$$

The same conclusion should be true also in dimensional regularization used in [14] (see also Appendix A) where $V_d = \pi^{\frac{d-1}{2}} \Gamma(-\frac{d-1}{2})$ contains a pole for odd $d \rightarrow d - \varepsilon$ (i.e. $\frac{1}{\varepsilon} \sim \ln R$). Here the product $\bar{\zeta}_{\Delta_s}(z) V_d$ in (2.14) has also a non-trivial finite part whose s -dependent coefficient need not be the same as the coefficient of the $\frac{1}{\varepsilon}$ pole part. However, since $\sum_s \bar{\zeta}'_{\text{eff},s}(0) = 0$ property was shown in [8] to be true for any d , the multiplication by s -independent factor V_d should not change this conclusion.

⁵For odd d there are no UV divergences, i.e. one automatically has $\zeta_{\Delta_s}(0) = 0$, but one is still to show that the finite part vanishes too, i.e. $\zeta'_{\text{tot}}(0) = 0$.

In general, if one uses dimensional (e.g. proper-time) UV cutoff L , then $\ln \det \Delta_s$ in (2.14) will contain also power divergences. The coefficients of such L^{d-2n} terms in d dimensions are controlled by the Seeley coefficients or $\zeta_{\Delta_s}(0)$ functions in $d' = d - 2n$ dimensions. Since the sum over s of the corresponding combination of $\zeta(0)$ vanishes in any $d > 2$ [8], this suggests that all power divergences should thus be absent too, demonstrating $Z_{\text{tot}} = 1$ in the proper-time cutoff regularization. This cancellation of power divergences is again analogous to what happens in supersymmetric theories.

For example, in $d = 4$ there will be L^4 divergence with the coefficient of the total number of degrees of freedom (which vanishes according to (2.4),(2.6)) and also L^2 divergence proportional to $\text{tr}(\frac{1}{6}R - M^2)$ for an operator $-\nabla^2 + M^2$ (here $R = 12\sigma a^{-2}$ is the scalar curvature of the background metric). The coefficient of the R -term is again the total number of degrees of freedom, while the contribution of the M^2 terms is found to be (we suppress the overall sign factor σ)

$$2 + \sum_{s=1}^{\infty} (M_{s,s-1}^2 N_s^\perp - M_{s-1,s}^2 N_{s-1}^\perp) = 2 + 4 \sum_{s=1}^{\infty} (1 + 2s^2) = 0, \quad (2.17)$$

where in the last step we used the standard zeta-function regularization equivalent to $e^{-\epsilon s}$ cutoff. In any even d one finds the same vanishing result using the regularization factor $f(s, \epsilon) = \exp[-\epsilon(s + \frac{d-4}{2})]$ as in [8].

Explicitly, higher Seeley coefficients for an operator $-\nabla^2 + M^2$ on a constant curvature space are given by $\sum_{k,n} \text{tr}(R^k M^{2n})$ and thus are expressed in terms of terms proportional to $\text{tr}(M^{2n})$. Eq. (2.17) is interpreted as $\text{tr} M^2 = 0$; one can show also the validity of its higher power analogs or mass sum rules $\text{tr}(M^{2n}) = 0$, where

$$\text{tr}(M^{2n}) = [2(d-3)]^n + \sum_{s=1}^{\infty} e^{-\epsilon(s + \frac{d-4}{2})} \left[M_{s,s-1}^{2n} N_s + M_{s+2,s+1}^{2n} N_{s-2} - 2M_{s-1,s}^{2n} N_{s-1} \right]. \quad (2.18)$$

Here the first term is the contribution from the spin 0 field.⁶

It is natural to conjecture that the $Z_{\text{tot}} = 1$ property (2.16) should be true not only for the 1-loop partition function, but also for the exact AdS_d vacuum partition function of the Vasiliev theory (i.e. at any quantum loop order in the $1/N$ coupling expansion). As already mentioned, this is the requirement of the vectorial AdS/CFT duality: the logarithm of the partition function of the dual free $U(N)$ scalar theory has only the order N term (that should match the vacuum value of the classical action of the Vasiliev theory). This further strengthens the analogy with a supersymmetric or topological quantum field theory.

Finally, let us note that the property (2.16) need not apply to quotients of the AdS_d space – for example, the MHS partition function on thermal quotient of AdS_d is non-trivial (see [9] and refs. there).

⁶For example, for $d = 4$, $\text{tr}(M^4) = 4 - 2 \sum_{s=1}^{\infty} e^{-\epsilon s} (s^2 - 4)(5s^2 + 1) = -\frac{240}{\epsilon^5} + \frac{76}{\epsilon^3} + \frac{8}{\epsilon} + \mathcal{O}(\epsilon)$, $\text{tr}(M^6) = 8 + 2 \sum_{s=1}^{\infty} e^{-\epsilon s} (2s^6 - 23s^4 + 67s^2 + 8) = \frac{2880}{\epsilon^7} - \frac{1104}{\epsilon^5} + \frac{268}{\epsilon^3} + \frac{16}{\epsilon} + \mathcal{O}(\epsilon)$. In $d = 4$ the exponential regularization is equivalent to the use of the Riemann zeta function prescription with $\zeta_R(-2n) = 0$, $\zeta_R(0) = -\frac{1}{2}$. For general n the summand can be written as $h(s) + h(-s)$, where $h(s) = M_{s,s-1}^{2n} N_s - M_{s-1,s}^{2n} N_{s-1}$, $h(-s) = M_{s+2,s+1}^{2n} N_{s-2} - M_{s-1,s}^{2n} N_{s-1}$. Then the only non-vanishing contribution in the sum over s is coming from the s^0 term, giving $2f(0)\zeta_R(0) = -2^n$, and this cancels against the first term in (2.18).

2.2 Conformal higher spins

2.2.1 Flat space

Let us now consider the free partition function for conformal higher spin (CHS) theory [12, 7]. The flat space action for a free CHS field in d dimensions is $\int d^d x \varphi_s P_s \partial^{2s+d-4} \varphi_s$, where P_s is projector to transverse traceless totally symmetric rank s field. Here $s = 0$ is a non-dynamical scalar, $s = 1$ is the Maxwell vector, $s = 2$ is the Weyl graviton, etc. The corresponding partition function in $d = 4$ is [7]

$$Z_{\text{CHS},s} = \left[\frac{(\det \Delta_{s-1})^{s+1}}{(\det \Delta_s)^s} \right]^{1/2} = \prod_{k=0}^{s-1} \left[\frac{\det \Delta_{k+1}}{\det \Delta_k} \right]^{1/2}, \quad (2.19)$$

where as in (2.1) the operator $\Delta_s = -\partial^2$ is defined on symmetric traceless tensors.

CHS fields having dimension $2 - s$ are sources or "shadow fields" for spin s conserved bilinear currents $J_s(\phi)$ built out of a free $U(N)$ scalar field; they are also boundary values for the corresponding dual MHS theory in AdS_{d+1} . An interacting CHS theory may be defined as an induced one [24–26], obtained by integrating out ϕ in the path integral defined by the action $\int d^4 x [\partial \phi^* \partial \phi + \sum_s J_s(\phi) \varphi_s]$. The resulting interacting CHS theory contains all fields with spins $s = 0, 1, 2, \dots$. The corresponding free partition function in flat background is given by

$$(Z_{\text{CHS}})_{\text{tot}} = \prod_{s=1}^{\infty} Z_{\text{CHS},s} = \left[\frac{\det \Delta_0}{\det \Delta_1} \right]^{1/2} \left[\frac{(\det \Delta_1)^3}{(\det \Delta_2)^2} \right]^{1/2} \left[\frac{(\det \Delta_2)^4}{(\det \Delta_3)^3} \right]^{1/2} \dots \quad (2.20)$$

Formally cancelling similar factors in the numerator and the denominator of (2.20) leads, in contrast to (2.2), to a non-trivial result

$$(Z_{\text{CHS}})_{\text{tot}} \rightarrow (Z_{\text{CHS}})'_{\text{tot}} = \prod_{s=0}^{\infty} \det \Delta_s. \quad (2.21)$$

This rearrangement of an infinite product in (2.20) effectively corresponds to its particular regularization.

Alternatively, we may use that each $Z_{\text{CHS},s}$ factor (2.19) may be written as in (2.3), i.e. as $(Z_0)^{\nu_s} = [\det \Delta_0]^{-\nu_s/2}$ where here $\nu_s = s(s+1)$ is the number of dynamical degrees of freedom of a CHS field in $d = 4$. Then

$$(Z_{\text{CHS}})_{\text{tot}} = \prod_{s=0}^{\infty} (Z_0)^{\nu_s} = (Z_0)^{\nu_{\text{tot}}}, \quad \nu_{\text{tot}} = \sum_{s=0}^{\infty} \nu_s, \quad \nu_s = s(s+1). \quad (2.22)$$

The total number of CHS degrees of freedom vanishes if one uses the regularization suggested by the relation to the MHS theory in AdS_d with $d = d + 1$ (in which also the total conformal anomaly vanishes [5, 7, 8]). Indeed, doing the sum in (2.22) with the $\exp[-\epsilon(s + \frac{d-3}{2})]$ cutoff as in (2.6) we get in the $d = 4$ case

$$\nu_{\text{tot}} = \sum_{s=0}^{\infty} s(s+1) e^{-\epsilon(s+\frac{1}{2})} \Big|_{\text{fin.}} = 0, \quad \text{i.e.} \quad (Z_{\text{CHS}})_{\text{tot}} = 1, \quad (2.23)$$

as in the MHS case in (2.2),(2.6). This conclusion generalises to the case of the CHS theory in d even dimensions where the partition function is [7]

$$Z_{\text{CHS},s} = \left[\left(\frac{1}{\det \Delta_{s\perp}} \right)^{\frac{d-4}{2}} \prod_{k=0}^{s-1} \frac{\det \Delta_{k\perp}}{\det \Delta_{s\perp}} \right]^{1/2} = (Z_0)^{\nu_s}, \quad (2.24)$$

$$\nu_s = \frac{(d-3)(2s+d-4)(2s+d-2)(s+d-4)!}{2(d-2)!s!}. \quad (2.25)$$

Then $\nu_{\text{tot}} = \sum_{s=0}^{\infty} \nu_s e^{-\epsilon(s+\frac{d-3}{2})} \Big|_{\text{fin.}} = 0$ and thus $(Z_{\text{CHS}})_{\text{tot}} = 1$. Notice that here d is the dimension of the boundary where CHS theory is defined; the related MHS theory defined in $d = d + 1$ has equivalent regularization used, e.g., in (2.6). This regularization should be the one that is consistent with the symmetries of the CHS theory.

At the same time, if we start with the rearranged product (2.21) that can be written as

$$(Z_{\text{CHS}})'_{\text{tot}} = (Z_0)^{-2N_{\text{tot}}}, \quad N_{\text{tot}} = \sum_{s=0}^{\infty} N_s, \quad N_s = (s+1)^2, \quad (2.26)$$

we find that $N_{\text{tot}} = \frac{1}{24}$ if one uses the same regularization as in (2.23).⁷ This illustrates an ambiguity associated with formal rearrangements of an infinite product: the result depends on a regularization.

2.2.2 Ricci-flat space

In contrast to the 2-derivative massless higher spin theory the CHS theory is expected to admit a Ricci-flat (or, more generally, Bach) background as its classical solution and each CHS field should have proper gauge invariance in such background. Thus the free CHS partition function in $R_{mn} = 0$ should be well-defined (gauge-independent). If one makes a bold conjecture (known to be true for $s = 1, 2$)⁸ that the $2s$ derivative covariant CHS operator can be factorized into a product of standard 2-derivative spin s Lichnerowicz operators (2.7) [7]:

$$Z_{\text{CHS},s} = \left[\frac{(\det \Delta_{Ls-1})^{s+1}}{(\det \Delta_{Ls})^s} \right]^{1/2}. \quad (2.27)$$

The same rearrangement of the infinite product as in (2.20) then gives the following Ricci-flat space generalization of (2.21)

$$(Z_{\text{CHS}})_{\text{tot}} = \prod_{s=1}^{\infty} Z_{\text{CHS},s} \rightarrow (Z_{\text{CHS}})'_{\text{tot}} = \prod_{s=0}^{\infty} \det \Delta_{Ls}. \quad (2.28)$$

From (2.27) one finds, in particular, the following expression for the $\beta_1 = c$ – a conformal anomaly coefficient (coefficient of the R^*R^* term in trace anomaly on Ricci-flat background) [7]:

$$c_s - a_s = \frac{1}{720} \nu_s (4 - 45\nu_s + 15\nu_s^2), \quad \nu_s = s(s+1). \quad (2.29)$$

Then $\sum_{s=1}^{\infty} (c_s - a_s) = 0$ in the same regularization [8] as in (2.23),

⁷ N_{tot} does vanish in a different regularization: with the cutoff factor being $e^{-\epsilon(s+1)}$.

⁸ This conjecture is likely to be wrong for $s = 3$ [27] unless the background curvature tensor is subject to further constraints, but may be true as far as one is allowed to ignore terms with covariant derivatives of the curvature (which, for example, do not contribute to nontrivial part of the conformal anomaly in $d = 4$).

2.2.3 Conformally-flat space: S^4

As was shown in [5, 7, 8], the sum of the conformal anomaly a_s -coefficients over all s vanishes (this is true, in particular, in the same regularization as used in (2.23)). One might then expect that the total partition function of the CHS theory on a conformally-flat space should be simply related to the one in flat space. For example, $Z_{\text{tot}}(S^4)$ with the product over s computed with the regularization in (2.23) may again be equal to 1.

This would be consistent with the relation between the spin s MHS partition function in AdS_5 and the corresponding CHS partition function on S^4 [5, 7, 11] (see also [28, 29])

$$Z_{\text{CHS},s}(S^4) = \frac{Z_{\text{MHS},s}^-(AdS_5)}{Z_{\text{MHS},s}^+(AdS_5)}. \quad (2.30)$$

Here $Z_{\text{MHS},s}^+$ is the massless higher spin partition function with the standard (Dirichlet) b.c. (as assumed in (2.12)) while $Z_{\text{MHS},s}^-$ is its alternative (Neumann) b.c. counterpart. The relation (2.30) should be true for any $s = 0, 1, 2, \dots$ and should thus also apply also to the total products over s . The computation of the spectral zeta-function in [8] (in which the infinite AdS_5 volume was assumed to factorize uniformly) implies (2.16), i.e. $(Z_{\text{MHS}}^+(AdS_5))_{\text{tot}} = 1$ and also $(Z_{\text{MHS}}^-(AdS_5))_{\text{tot}} = 1$.⁹ Equivalently, these properties hold in the same regularization of the sum over spins as used in (2.23). Assuming the validity of (2.30) one should then expect to find that in this summation prescription

$$(Z_{\text{CHS}})_{\text{tot}}(S^4) = \prod_{s=1}^{\infty} Z_{\text{CHS},s}(S^4) = 1. \quad (2.31)$$

The verification of (2.30) by directly computing the determinants appearing on the both sides of the equality turns out to be non-trivial. The expression for $Z_{\text{CHS},s}(S^4)$ depends on a choice of UV regularization, while $Z_{\text{MHS},s}^{\pm}(AdS_5)$ depends on a choice of IR regularization, and these regularizations should be properly coordinated for (2.30) to hold. The question of how to do this was previously addressed only in the spin 0 analog of the relation (2.30) in [14, 30] where $Z_{\text{CHS},s}$ is replaced by the partition function of the order $2r$ GJMS operator [31] and $Z_{\text{MHS},s}$ – by the AdS_5 partition function of the massive scalar with $m^2 = \Delta(\Delta - 4) = r^2 - 4$. We shall discuss this case in detail Appendix A.1 below. For $s > 0$ the relation (2.30) was verified for the leading singular (logarithmically divergent) parts only: the spin-dependent coefficient [5] of the IR divergent $\ln R$ term (cf. (2.15)) in $\ln \frac{Z_{\text{MHS},s}^-(AdS_5)}{Z_{\text{MHS},s}^+(AdS_5)}$ indeed matches the (conformal anomaly a_s) coefficient [7] of the UV divergent $\ln \Lambda$ term in $\ln Z_{\text{CHS},s}(S^4)$.¹⁰

If one uses an IR regularization in AdS_5 in which the volume universally factorizes as in (2.14) then the coefficients of the $\ln R$ and finite parts in the MHS side of (2.30) appear to have the same spin dependence. This is certainly not so a priori for the coefficients

⁹The condition $\zeta_{\text{MHS}}(0) = 0$ is automatic in the AdS_5 case, while $\zeta'_{\text{MHS}}(0) = 0$ is valid for both choices of the boundary conditions.

¹⁰Here Λ stands for UV cutoff in the $d = 4$ CHS theory where determinants have similar expression as in (2.14).

of the UV divergent and finite parts on the CHS side of (2.30).¹¹ As was pointed out in the $s = 0$ case in [14], to be able to systematically match the finite parts of the partition functions on S^d and AdS_{d+1} one may use dimensional regularization, i.e. $d \rightarrow d - \varepsilon$. Then for even d the regularized AdS_{d+1} volume $\pi^{\frac{d}{2}} \Gamma(-\frac{d}{2})$ will have $\frac{1}{\varepsilon}$ pole term that may hit an order ε term in $\bar{\zeta}_{\Delta_s}$ in (2.14) to produce a non-trivial finite contribution that may match the finite term present in the S^d partition function. Generalizing the dimensional regularization approach of [14] to spin $s > 0$ case on AdS_5 side, in Appendix A.2 we shall demonstrate the matching of the most non-trivial transcendental finite parts of the CHS and MHS sides of (2.30). We shall then provide a check of the validity of the $(Z_{\text{CHS}})_{\text{tot}} = 1$ property (2.31) in this regularization consistent with (2.30) using the same summation over spins prescription as in (2.23).

Explicitly, the CHS partition function on S^4 is given by the following generalization of (2.19) (cf. (2.10),(2.11),(2.12)) [7]

$$Z_{\text{CHS},s}(S^4) = \prod_{k=0}^{s-1} Z_{s,k}, \quad Z_{s,k} = \left[\frac{\det \Delta_{k \perp}(M_{k,s}^2)}{\det \Delta_{s \perp}(M_{s,k}^2)} \right]^{1/2}, \quad M_{s,k}^2 \Big|_{d=4} = s - (k-1)(k+2). \quad (2.32)$$

For the corresponding conformal anomaly a-coefficient one finds

$$a_s = \frac{1}{720} \nu_s (3\nu_s + 14\nu_s^2), \quad \nu_s = s(s+1), \quad (2.33)$$

and then $\sum_{s=1}^{\infty} a_s = 0$ in the same regularization as in (2.23).¹² Using the known spectra of the second-order Laplace operators on S^4 one may also compute explicitly the finite part of the determinants in (2.32)¹³ and match the CHS expression with the MHS one in (2.30) (see Appendix A.2).

An analog of (2.31) may be expected to be true for any conformally-flat space, e.g., for $\mathbb{R} \times S^3$ (indeed, the vanishing of the CHS Casimir energy on S^3 was demonstrated in [10]). At the same time, this need not apply to orbifolds of conformally-flat space: for example, the finite temperature CHS partition function on $S^1_\beta \times S^3$ is non-trivial [10] (and is equal to the ratio (2.30) of the non-trivial MHS partition functions on thermal quotient of AdS_5).¹⁴

¹¹One may of course formally absorb the finite part into a redefinition of UV cutoff $\Lambda \rightarrow \tilde{\Lambda}$ but that will make $\tilde{\Lambda}$ spin-dependent, precluding its identification with R on the AdS_5 side and also complicating the issue of summation over s .

¹²Together with the vanishing of the sum of $c_s - a_s$ in (2.29) this implies the one-loop UV finiteness of the CHS theory on a curved 4d background.

¹³We shall ignore the (rational) contribution of the multiplicative anomaly, i.e. the ratio of $\det(O_1 \dots O_n)$ to $\det O_1 \dots \det O_n$ [32]. The role of this anomaly in CHS theory is unclear: it does not appear in the formulation based on introducing auxiliary fields to write the CHS action in terms of second derivative operators only [33]. It is possible that ignoring this anomaly is required for consistency with the ratio of the MHS partition functions in (2.30).

¹⁴While $S^1_\beta \times S^3$ is locally conformally flat (the Weyl tensor vanishes) this is not true globally: the periodicity of S^1 coordinate is important. Indeed, to show that $\mathbb{R} \times S^3$ is conformal to flat space one uses that $dx^2 + dS^3 = y^{-2}(dy^2 + y^2 dS^3) = y^{-2} dy_n dy_n$. If x is an angle variable we cannot set it equal to $\ln y$ globally.

3 2-derivative conformal symmetric tensor theory

Let us now consider another family of "higher-spin" (symmetric traceless tensor) actions [13] in 4 dimensions which, like the CHS ones are covariant under Weyl rescaling of the background metric but like the MHS ones (which are not conformal for $s > 1$) are only second order in derivatives. We shall refer to this theory as "conformal symmetric tensor" (CST) theory.

These CST fields are non-unitary lacking full higher spin gauge invariance: even in (conformally) flat space limit they have only scalar gauge invariance. In fact, they can be interpreted as "maximal depth" (minimal gauge invariance) representatives of a family of conformal fields (called "FT fields" in [16]) containing CHS fields as maximal gauge invariance members. Like CHS fields are associated with massless fields in AdS_5 , CST fields in 4d correspond to "maximal depth" partially massless fields in AdS_5 [16].¹⁵

Below we shall first introduce the corresponding bilinear action in curved background, then discuss in detail the spin 2 case, and finally consider the partition function for the general spin s case.

3.1 Classical Lagrangian

Given a rank s totally symmetric traceless tensor φ_s there are several options to construct an action invariant under Weyl rescalings of the metric combined with a rescaling of the field φ_s . If the action contains $2n$ derivatives, i.e. starts with

$$\int d^d x \sqrt{g} \varphi_s (\nabla^2)^n \varphi_s + \dots, \quad n = 1, 2, \dots, \quad (3.1)$$

then one should require the invariance under $(\Omega = \Omega(x))$

$$g'_{\mu\nu} = \Omega^2 g_{\mu\nu}, \quad \varphi'_{\mu_1 \dots \mu_s} = \Omega^\gamma \varphi_{\mu_1 \dots \mu_s}, \quad \gamma = s + n - \frac{1}{2}d. \quad (3.2)$$

¹⁵As discussed in [16], one may consider a generalize triple of families of fields (here $d = 4$):

(i) starting with higher-derivative conformal scalars in R^4 (higher-order singletons) with the action $\int d^4 x \phi_i^* (\partial^2)^\ell \phi_i$ one may construct dimension $\Delta = 3 + s - t$ partially conserved currents J_s of spin s and depth t (with $1 \leq t \leq s$), $\partial_{m_1} \dots \partial_{m_t} J^{m_1 \dots m_s} = 0$; the corresponding Verma module $V(\Delta, s) = V(3 + s - t, s)$ is reducible for $t \leq s$ and the irreducible module is $D(3 + s - t, s) = V(3 + s - t, s) / V(3 + s, s - t)$ [34, 35];

(ii) the sources for these currents or the corresponding "shadow" fields are primary conformal fields of depth t which are totally symmetric traceless tensors $\varphi_{m_1 \dots m_s}$ of dimension $\Delta = 1 + t - s$ with the action $\int d^4 x \varphi_s (\partial^2)^{1+s-t} \varphi_s$;

(iii) these are naturally associated to partially massless fields in AdS_5 with the action $\int d^5 x \phi_s (\nabla^2 + \dots) \phi_s$ having gauge invariance $\delta \phi_{\mu_1 \dots \mu_s} = \nabla_{\mu_1} \dots \nabla_{\mu_t} \epsilon_{\mu_{t+1} \dots \mu_s} + (g_{\mu\nu} - \text{terms})$; the fields ϕ_s are dual to $J_{m_1 \dots m_s}$ or have $\varphi_{m_1 \dots m_s}$ as boundary values.

The minimal depth $t = 1$ case corresponds to conserved currents, φ_s as CHS fields in $d = 4$ and ϕ_s as MHS fields in AdS_5 . In the maximal depth case $t = s$ which we will be considering below φ_s correspond to the CST fields while ϕ_s – to partially massless fields of maximal depth.

The field content of the AdS_5 theory dual to singlet sector of $U(N)$ $(\partial^2)^\ell$ scalar theory follows from the generalized Flato-Fronsdal theorem [16]: $D(2 - \ell, 0) \otimes D(2 - \ell, 0) = \oplus_{s=0}^\infty \oplus_{k=1}^\ell D(4 + s - 2k, s)$, where the sum goes over partially massless fields of different odd depths $t = 1, \dots, 2\ell - 1$. Thus in contrast to the MHS fields in the minimal depth case the maximal depth $t = s$ fields ϕ_s do not form a "closed" subset, i.e. one is to group together fields of different depth to get the dual AdS_5 theory.

For example, in the CHS case $n = s + \frac{1}{2}(d - 4)$ and $\gamma = 2s - 2$. The family of conformal operators with $n = 1, 2, \dots$ generalises the scalar GJMS [31] one to the $s > 0$ case. In the case of 2-derivative kinetic term ($n = 1$) corresponding to CST theory one has $\gamma = s - \frac{1}{2}(d - 2)$. The larger the n , the more gauge symmetries consistent with locality of the action one may expect to be realisable (at least, in the flat space limit). Indeed, in the CHS case one has maximal gauge symmetry with rank $s - 1$ tensor parameter while in the $n = 1$ case there will be only $\delta\varphi_s = \partial^s\sigma$ gauge symmetry with scalar gauge parameter.

Having less than maximal gauge symmetry, one will be unable to eliminate all the time-like field components (contributing to the action with negative sign) by a choice of a "unitary" gauge ($n > 1$ actions will be non-unitary also due to higher derivative kinetic term). While the standard 2-derivative Fronsdal massless (maximally gauge-invariant) higher spin action is unitary but not conformally invariant, the $n = 1$ conformally invariant CST action will not be unitary due to insufficient gauge symmetry.

The $n = 1$ action invariant under (3.2) was found in [13] (cf. also [36] for $d = 4$): it is given by $S_s = -\frac{1}{s!} \int d^d x \sqrt{g} \mathcal{L}_s$ where

$$\begin{aligned} \mathcal{L}_s(d) = & \nabla^\lambda \varphi^{\mu_1 \dots \mu_s} \nabla_\lambda \varphi_{\mu_1 \dots \mu_s} - \frac{4s}{2s+d-2} \nabla_\rho \varphi^{\mu_1 \dots \mu_{s-1} \rho} \nabla_\lambda \varphi_{\mu_1 \dots \mu_{s-1} \lambda} \\ & + \frac{2s}{d-2} R_{\rho\lambda} \varphi^{\mu_1 \dots \mu_{s-1} \rho} \varphi_{\mu_1 \dots \mu_{s-1} \lambda} - \frac{4s-d^2+4d-4}{4(d-1)(d-2)} R \varphi^{\mu_1 \dots \mu_s} \varphi_{\mu_1 \dots \mu_s} \\ & + \omega C_{\alpha\beta\rho\lambda} \varphi^{\mu_1 \dots \mu_{s-2} \alpha \rho} \varphi_{\mu_1 \dots \mu_{s-2} \beta \lambda}, \quad \mathcal{L}_{\text{CST},s} \equiv \mathcal{L}_s(d=4). \end{aligned} \quad (3.3)$$

Here C is the Weyl tensor and ω is an arbitrary constant.

In what follows we shall consider the $d = 4$ case. In this case the flat space limit of (3.3) is invariant under the scalar gauge transformations¹⁶

$$\mathcal{L}_{\text{CST},s} = \partial^\lambda \varphi^{\mu_1 \dots \mu_s} \partial_\lambda \varphi_{\mu_1 \dots \mu_s} - \frac{2s}{s+1} (\partial^\lambda \varphi_{\mu_1 \dots \mu_{s-1} \lambda})^2, \quad \delta\varphi_{\mu_1 \dots \mu_s} = \partial_{\mu_1} \dots \partial_{\mu_s} \sigma - \text{traces}. \quad (3.4)$$

The field φ_s has canonical dimension 1 and thus corresponds to a (non-unitary) representation $(1; \frac{s}{2}, \frac{s}{2})$ of $SO(2, 4)$.¹⁷ This representation is unitary ($\Delta \geq 2 + j_1 + j_2$) only for $s = 0, 1$. The gauge parameter σ corresponds to the $d = 4$ conformal group representation $(1 - s; 0, 0)$, so that (3.4) describes a "short" representation¹⁸

$$[1; \frac{s}{2}, \frac{s}{2}] = (1; \frac{s}{2}, \frac{s}{2}) - (1 - s; 0, 0). \quad (3.5)$$

In $d = 4$ and $s = 1$ (3.3) gives the standard Maxwell Lagrangian. Let us discuss in detail the spin 2 case.

3.2 Rank 2 case

In $d = 4$ the $s = 2$ case of (3.3) is

$$\begin{aligned} \mathcal{L}_{\text{CST},2} = & \nabla^\lambda \varphi^{\mu\nu} \nabla_\lambda \varphi_{\mu\nu} - \frac{4}{3} (\nabla_\mu \varphi^{\mu\nu})^2 + 2 R_{\rho\lambda} \varphi^{\mu\rho} \varphi_\mu^\lambda \\ & - \frac{1}{6} R \varphi^{\mu\nu} \varphi_{\mu\nu} + \omega C_{\mu\nu\rho\lambda} \varphi^{\mu\rho} \varphi^{\nu\lambda}. \end{aligned} \quad (3.6)$$

¹⁶ In general dimension d , the scalar gauge invariance is present in the case when the CST kinetic operator contains $2 + (d - 4)$ derivatives. Such conformal field (see [35, 15]) is the maximal depth FT field in [16].

¹⁷ We use the notation $(\Delta; j_1, j_2)$ where Δ is the scaling dimension or conformal weight and (j_1, j_2) are the $SU(2)$ weights of $SO(3, 1)$.

¹⁸ This case should correspond to a particular degenerate module of the conformal group [35, 15, 16].

For comparison, the quadratic term in the expansion of the Einstein action in generic background is (here $g_{\mu\nu} \rightarrow g_{\mu\nu} + h_{\mu\nu}$, $\varphi_{\mu\nu} = h_{\mu\nu} - \frac{1}{4}g_{\mu\nu}h$, $h = h^\mu_\mu$)

$$\begin{aligned} \mathcal{L}_E = & \nabla^\lambda \varphi^{\mu\nu} \nabla_\lambda \varphi_{\mu\nu} - 2 \left[\nabla_\mu (\varphi^{\mu\nu} - \frac{1}{2} g^{\mu\nu} h) \right]^2 + \frac{1}{4} h \nabla^2 h \\ & + \frac{5}{3} R \varphi^{\mu\nu} \varphi_{\mu\nu} - 2 C_{\mu\alpha\nu\beta} \varphi^{\mu\nu} \varphi^{\alpha\beta} \end{aligned} \quad (3.7)$$

The action for (3.7) has the standard (vector-parameter) gauge invariance on a $R_{\mu\nu} = 0$ background, while this is not the case for (3.6).

We shall refer to (3.6) as conformal spin 2 Lagrangian. Let us mention some earlier work related to it. As is well known, in contrast to massless spin 0 and spin 1 fields, the spin 2 Einstein graviton does not represent a conformal theory in flat space [37].¹⁹ The $SO(2,4)$ conformal group representation for the symmetric traceless tensor $\varphi_{\mu\nu}$ is $(\Delta; j_1, j_2) = (1; 1, 1)$. The corresponding free conformally invariant equations of motion [38] (see also [39, 40])

$$\partial^2 \varphi_{\mu\nu} - \frac{4}{3} \left(\partial_\alpha \partial_{(\mu} \varphi_{\nu)}^\alpha - \frac{1}{4} g_{\mu\nu} \partial_\alpha \partial_\beta \varphi^{\alpha\beta} \right) = 0 \quad (3.8)$$

follow from the flat-space Lagrangian (3.6) or (3.4) with scalar gauge invariance

$$L_{\text{CST},2} = \partial^\lambda \varphi^{\mu\nu} \partial_\lambda \varphi_{\mu\nu} - \frac{4}{3} (\partial_\mu \varphi^{\mu\nu})^2, \quad \delta \varphi_{\mu\nu} = (\partial_\mu \partial_\nu - \frac{1}{4} g_{\mu\nu} \partial^2) \sigma. \quad (3.9)$$

$\varphi_{\mu\nu}$ corresponds to $(1; 1, 1)$ conformal group representation which is non-unitary [41]. As already mentioned, non-unitarity is implied also by the absence of the maximal (vector-parameter) spin 2 gauge invariance: in fact, (3.8) describes a combination of spin 2 and two spin 1 massless on-shell fields with total $(9 - 1) - 2 = 6$ physical degrees of freedom. Splitting

$$\varphi_{\mu\nu} = \varphi_{\mu\nu}^\perp + \partial_{(\mu} V_{\nu)}^\perp + (\partial_\mu \partial_\nu - \frac{1}{4} g_{\mu\nu} \partial^2) \sigma \quad (3.10)$$

and accounting for the Jacobian of this transformation and the decoupling of σ one finds that the corresponding flat-space partition function that can be written as

$$Z_{\text{CST},2} = \left[\frac{(\det \Delta_0)^3}{\det \Delta_2} \right]^{1/2} = \left[\frac{\det \Delta_{1\perp}}{\det \Delta_{2\perp}} \right]^{1/2} \left[\left(\frac{\det \Delta_0}{\det \Delta_{1\perp}} \right)^2 \right]^{1/2}. \quad (3.11)$$

This is recognised to be a product of the standard massless spin 2 and the square of the massless spin 1 partition functions (cf. (2.1)).

A curved space generalization of the equations (3.8) was first attempted in [37], but there the transversality constraint $\nabla^\mu \varphi_{\mu\nu} = 0$ was added by hand (i.e. V_μ^\perp part was ignored) making the theory effectively non-local or non-Lagrangian. It was pointed out in [37] that the coefficient ω of the Weyl tensor coupling in the generalization of (3.8) is, in general, arbitrary, being allowed by the Weyl covariance condition. The consistent $d = 4$

¹⁹This follows, e.g., from the absence of Weyl invariance of the Einstein theory. In particular, the conformal weights do not match: according to (3.2), the conformal spin 2 field $\varphi_{\mu\nu}$ is to have Weyl weight 1 while the metric itself transforms with weight 2. For all massless spins $s \geq 2$ one has scale invariance but no special conformal invariance.

Lagrangian (3.6) first appeared for $\omega = 0$ in [20].²⁰ Its $\omega = -2$ version was found in [43].²¹ In its general form (with an arbitrary ω) the Lagrangian (3.6) first appeared in [13].

3.2.1 Scalar gauge invariance and partition function in Ricci-flat space

An important observation made in [20] is that restricted to a conformally flat homogeneous background (AdS_4 or S^4) the Lagrangian (3.6) admits an analog of the scalar gauge invariance in (3.9),

$$\delta\varphi_{\mu\nu} = (\nabla_\mu \nabla_\nu - \frac{1}{4} g_{\mu\nu} \nabla^2) \sigma. \quad (3.12)$$

This curved-space generalization of the scalar gauge invariance (3.9) is an important consistency requirement. As we shall explain in Appendix A, this invariance generalises also to the Einstein-space backgrounds *provided* the coefficient ω in (3.6) is chosen to be

$$\omega = -2, \quad (3.13)$$

i.e. is the same as in the expansion of the Einstein action (3.7) or in the Lichnerowicz operator.²² The presence of the scalar gauge invariance in a gravitational background indicates that one may be able to couple this conformal rank 2 tensor field to Einstein gravity in a consistent way (cf. [47]).

This choice is special also for another (related) reason: in this case the differential operator in (3.6) factorises on an Einstein background: the spin 2 and spin 1 transverse parts in the curved space analog of the decomposition (3.10) decouple in the action. This makes the ∇^2 kinetic operator effectively diagonal and allows one to write the curved space generalization of the partition function (3.11) in a simple form in terms of determinants of the standard Lichnerowicz spin $s = 2, 1, 0$ operators.

Indeed, let us consider first the Ricci-flat background. Using the covariant version of the decomposition (3.10)

$$\varphi_{\mu\nu} = \varphi_{\mu\nu}^\perp + \nabla_{(\mu} V_{\nu)}^\perp + (\nabla_\mu \nabla_\nu - \frac{1}{4} g_{\mu\nu} \nabla^2) \sigma \quad (3.14)$$

in (3.6) with $R_{\mu\nu} = 0$ and $\omega = -2$ we get

$$\begin{aligned} \mathcal{L}_{\text{CST},2} &= \nabla^\lambda \varphi^{\mu\nu} \nabla_\lambda \varphi_{\mu\nu} - \frac{4}{3} (\nabla_\mu \varphi^{\mu\nu})^2 - 2 C_{\mu\nu\rho\lambda} \varphi^{\mu\rho} \varphi^{\nu\lambda} \\ &= \varphi_{\mu\nu}^\perp \Delta_{\text{L}2} \varphi^{\perp\mu\nu} + \frac{2}{3} V_\mu^\perp (\Delta_{\text{L}1})^2 V^{\perp\mu}, \end{aligned} \quad (3.15)$$

²⁰In general dimension d , its Weyl invariant 2-derivative analog found in [42] lacks scalar gauge invariance for $d > 4$.

²¹The sign of the curvature in [20, 42, 43] was opposite to the conventional one we use here: $R^\mu{}_{\nu\rho\sigma} = \partial_\rho \Gamma_{\nu\sigma}^\mu + \dots$. Note also that ref.[44] contained (up to a misprint in Eq. (27)) the conformal spin 2 operator that is the same as in [20]. Finally, there were also attempts to define the conformally invariant operator on a trace-full rank 2 tensor $h_{\mu\nu}$ [45, 46] which we will not consider here as we are interested in irreducible representations of the Lorentz group.

²²Note that ref.[13] considered a different special value: $\omega = -1$ in $d = 4$. For this choice, the Lagrangian (3.6) may be written in a factorized form

$$\mathcal{L} = \frac{2}{3} (\mathcal{D}\varphi)^{\mu\nu\lambda} (\mathcal{D}\varphi)_{\mu\nu\lambda}, \quad (\mathcal{D}\varphi)_{\mu\nu\lambda} = \nabla_\lambda \varphi_{\mu\nu} - \nabla_{(\mu} \varphi_{\nu)\lambda} - \frac{1}{3} (g_{\lambda(\mu} \nabla^{\rho} \varphi_{\nu)\rho} - 2 g_{\mu\nu} \nabla^{\rho} \varphi_{\lambda\rho}).$$

The significance of this observation is unclear, given that the absence of scalar gauge invariance in this case.

where Δ_{Ls} are the Lichnerowicz operators (2.7) on Ricci flat background. As was mentioned above, the decoupling of σ (which is a manifestation of the scalar gauge invariance) and separation of $\varphi_{\mu\nu}^\perp$ and V_μ^\perp is the consequence of the choice of ω in (3.13).²³

The Jacobian of the transformation in (3.14) is $J = [\det \Delta_{L1\perp} (\det \Delta_0)^2]^{1/2}$, so we get the following generalization of the flat-space expression (3.11)

$$Z_{\text{CST},2} = \left[\frac{(\det \Delta_0)^2}{\det \Delta_{L2\perp} \det \Delta_{L1\perp}} \right]^{1/2} = \left[\frac{(\det \Delta_0)^3}{\det \Delta_{L2}} \right]^{1/2}, \quad (3.16)$$

where $(\Delta_{L2})_{\mu\nu,\alpha\beta} = -g_{\mu(\alpha} g_{\beta)\nu} \nabla^2 - 2C_{\mu\alpha\nu\beta}$, $(\Delta_{L1})_{\mu\nu} = -g_{\mu\nu} \nabla^2$ and $\Delta_0 = -\nabla^2$.

3.2.2 Partition function on S^4

Let us now compute the partition function corresponding to the action (3.6) on conformally flat Einstein background, e.g., S^4 . Using (3.14) we find that σ decouples (manifesting scalar gauge invariance) and the dependence φ^\perp and V^\perp separates (we consider unit-radius S^4 , i.e. $R = 12$)

$$\mathcal{L}_{\text{CST},2}(S^4) = \varphi_{\mu\nu}^\perp \Delta_{2\perp}(4) \varphi^{\perp\mu\nu} + \frac{2}{3} V_\mu^\perp \Delta_{1\perp}(3) \Delta_{1\perp}(-3) V^{\perp\mu}. \quad (3.17)$$

As in (2.10)–(2.12) here $\Delta_{s\perp}(M^2) = -\nabla^2 + M^2$, acting in transverse symmetric traceless tensors of rank s . Taking into account the Jacobian of the transformation (3.14) (see, e.g., (A.8) in [7])

$$J = \left[\det \Delta_{1\perp}(-3) \det \Delta_0(-4) \det \Delta_0(0) \right]^{1/2}, \quad (3.18)$$

we end with the following partition function (generalising (3.11) in the flat-space case where all mass terms are zero)

$$Z_{\text{CST},2} = Z_{2,0} Z_{1,0}, \quad Z_{1,0} = \left[\frac{\det \Delta_0(0)}{\det \Delta_{1\perp}(3)} \right]^{1/2} = \left[\frac{(\det \Delta_0(0))^2}{\det \Delta_1(3)} \right]^{1/2}, \quad (3.19)$$

$$Z_{2,0} = \left[\frac{\det \Delta_0(-4)}{\det \Delta_{2\perp}(4)} \right]^{1/2} = \left[\frac{\det \Delta_0(-4) \det \Delta_1(-1)}{\det \Delta_2(4)} \right]^{1/2}. \quad (3.20)$$

Here $Z_{n,k}$ are defined as in (2.10),(2.11), i.e. $Z_{1,0}$ is the standard Maxwell partition function and $Z_{2,0}$ is the partition function corresponding to the $s = 2$ partially massless field [20]. $Z_{2,0}$ appears, together with the massless spin 2 (Einstein graviton) partition function $Z_{2,1}$, in the Weyl graviton partition function $Z_{\text{CHS},2} = Z_{2,1} Z_{2,0}$ which is a special case of (2.32) (see [7]). Thus we have the following relation

$$Z_{\text{CST},2} = \frac{Z_{\text{CHS},2} Z_{1,0}}{Z_{2,1}}. \quad (3.21)$$

In Appendix B we shall also present the expression for the finite temperature partition function on $S_\beta^1 \times S^3$ and discuss its interpretation in terms of counting of conformal operators in a spin 2 CFT in \mathbb{R}^4 corresponding to "shortened" representation with shadow counterpart as $(3;1,1) - (5;0,0)$.

²³The reason for this can be understood by comparing to the case of the Einstein theory (3.7) on $R_{\mu\nu} = 0$ background where the presence of the $\omega = -2$ Lichnerowicz operator is directly related to the vector-parameter gauge invariance, i.e. to the decoupling of the "longitudinal" part of $h_{\mu\nu}$ before gauge fixing.

3.2.3 Conformal anomaly coefficients and AdS/CFT interpretation

In $d = 4$ the conformal anomaly coefficients a and c in

$$b_4 = \beta_1 R^* R^* + \beta_2 (R_{\mu\nu}^2 - \frac{1}{3} R^2) = -a R^* R^* + c C^2, \quad \beta_1 = c - a, \quad \beta_2 = 2c, \quad (3.22)$$

can be found by doing two separate computations: (i) on conformally-flat space like S^4 (finding a coefficient) and (ii) on a Ricci-flat space (finding $c - a$ coefficient). The a coefficient corresponding to (3.6) thus does not depend on ω and is readily obtained from the expression for the partition function in (3.19). Using that [7]

$$\begin{aligned} a[\Delta_s(M^2)] &= \frac{1}{144}(s+1)^2 \left[(s+1)^2 - 3M^4 + 12M^2 - \frac{63}{5} \right], \\ a[\Delta_{\perp s}(M^2)] &= \frac{1}{720}(2s+1) \left[30s^3 + 85s^2 + 10s - 58 - 30(s^2 - 2)M^2 - 15M^4 \right], \end{aligned} \quad (3.23)$$

we get

$$a_{\text{CST},2} = a_{2,0} + a_{1,0} = \frac{53}{45} + \frac{31}{180} = \frac{27}{20}. \quad (3.24)$$

At the same time, computing the conformal anomaly c coefficient for the action in (3.6) on a Ricci-flat background is, in general, a non-trivial problem. The reason is that the corresponding second order operator is a non-minimal one, i.e. has "non-diagonal" highest derivative part (cf. [48]). This problem is, however, readily solved in the special $\omega = -2$ case (3.13) where one finds the explicit factorized expression for the partition function (3.16). Using that [17, 7]

$$\beta_1[\Delta_{Ls}] = \frac{1}{720}(s+1)^2 [21 - 20(s+1)^2 + 3(s+1)^4], \quad (3.25)$$

we find

$$(\beta_1)_{\text{CST},2} = \frac{31}{30}, \quad \text{i.e.} \quad c_{\text{CST},2} = \frac{143}{60}. \quad (3.26)$$

Let us now explain how these results fit into the general expressions for the conformal anomaly coefficients of massive $SO(2,4)$ representations discussed in [11]. Given a 4d conformal field in representation $(\Delta'; j_1, j_2)$ the corresponding shadow field $(\Delta; j_1, j_2)$ with $\Delta = 4 - \Delta'$ is associated (as a boundary value) to a massive higher spin in AdS_5 with mass $m^2 = (\Delta - 2)^2 - s^2$, $s = j_1 + j_2$ [49] (the corresponding kinetic operator for $j_1 \geq j_2$ is $-\nabla^2 + \Delta(\Delta - 4) - 2j_1$). A non-unitary representation field in 4d corresponds to a "partially massless" field in 5d. The CST field $(1; 1, 1)$ is thus related to $(3; 1, 1)$ spin 2 field in AdS_5 . In the case of $(\Delta; \frac{s}{2}, \frac{s}{2})$ massive field one finds via AdS_5 computation [5] (as in [11] we use hat to distinguish a long representation from shortened one)

$$\hat{a}(\Delta; \frac{s}{2}, \frac{s}{2}) = \frac{1}{720}(s+1)^2(\Delta - 2)^3(-3\Delta^2 + 12\Delta + 5s^2 + 10s - 7). \quad (3.27)$$

In the present case, accounting for the scalar gauge invariance, we should get (see (3.5); here $\Delta' = 1$ for the rank 2 field and $\Delta' = -1$ for the scalar gauge parameter field)

$$a_{\text{CST},2} = \hat{a}(3; 1, 1) - \hat{a}(5; 0, 0) = \frac{27}{20}. \quad (3.28)$$

This is indeed in agreement with direct computation in (3.24).

The expression for $\beta_1 = c$ – a coefficient for a conformal field associated to a representation $(\Delta; \frac{s}{2}, \frac{s}{2})$ field in AdS_5 suggested in [11] was

$$\hat{\beta}_1(\Delta; \frac{s}{2}, \frac{s}{2}) = \frac{1}{720}(s+1)^2(\Delta-2) \left[-3(\Delta-2)^4 - 5(s^2+2s-3)(\Delta-2)^2 + 8s^3 + 2s^2 - 12s - 8 \right] \quad (3.29)$$

Then using also (3.28) we get

$$c_{\text{CST},2} = \hat{c}(3;1,1) - \hat{c}(5;0,0) = \frac{143}{60} , \quad (3.30)$$

in agreement with (3.26). The equivalent result is found from the expression for β_1 proposed in [50, 51]

$$\hat{\beta}'_1(\Delta; \frac{s}{2}, \frac{s}{2}) = \frac{1}{180}(s+1)^2(\Delta-2) \left(1 + \frac{1}{4}s(s+2)[3s(s+2) - 14] \right) , \quad (3.31)$$

i.e. the present $s = 2$ case does not distinguish between the two expressions for β_1 or c discussed in [11].

3.3 General rank s case

Let us now consider the general s case with $d = 4$ Lagrangian $\mathcal{L}_{\text{CST},s}$ in (3.3). It turns out that for $s > 2$ the flat space scalar gauge invariance in (3.4) survives in curved space action (3.3) only in the conformally-flat case $C_{\mu\nu\lambda\rho} = 0$,²⁴ i.e. in contrast to the $s \leq 2$ cases it is absent in the Ricci flat background for any value of ω .²⁵

Also, the Weyl-covariant differential operator in (3.3) does not factorize, i.e. is non-minimal (has non-diagonal highest-derivative part in the transverse decomposition like (3.10)) unless the space is conformally flat. That makes it hard to compute the corresponding conformal anomaly c -coefficient.

Starting with the flat space case, the partition function corresponding to (3.4) is (cf. (2.1),(2.19))

$$Z_{\text{CST},s} = \left[\frac{(\det \Delta_0)^{s+1}}{\det \Delta_s} \right]^{1/2} = \prod_{k=1}^s \left[\frac{\det \Delta_0}{\det \Delta_{k\perp}} \right]^{1/2} = (Z_0)^{\nu_s} , \quad (3.32)$$

where the corresponding number of dynamical degrees of freedom is

$$\nu_s = N_s - (s+1) = \left[\frac{(2s+d-2)(s+d-3)!}{(d-2)!s!} - s - 1 \right]_{d=4} = s(s+1) . \quad (3.33)$$

²⁴Scalar gauge invariance is expected on a four dimensional conformally flat background as it is present in flat space and the action in curved background is Weyl invariant (for a related observation for partially massless fields in dS space see [52]).

²⁵One may draw an analogy with the standard massless spin s field with ∂^2 Lagrangian. In flat space one has maximal gauge invariance with spin $s-1$ parameter. This gauge invariance generalises to curved space in the case of conformally-flat homogeneous background. However, massless spin $s > 2$ quadratic actions do not admit consistent gauge-invariant generalizations to Ricci flat backgrounds. This is repaired in Fradkin-Vasiliev type theory where one adds all spins together and cancels variation under gauge invariance by variation of graviton and other spins. No such consistent interacting theory is known in the present second derivative conformal higher spin case (3.3), but one may conjecture that it may exist.

Thus the number of degrees of freedom of CST field happens to be the same as for CHS field of the same spin (2.22) and thus the corresponding flat-space partition functions match. As a result,

$$(Z_{\text{CST}})_{\text{tot}} = \prod_{s=1} Z_{\text{CST},s} = 1, \quad (3.34)$$

assuming one uses the same regularization as in (2.23).

Counting conformal gauge-invariant operators corresponding to the theory (3.4) one should be able to find the corresponding one-particle partition function generalising the $s = 2$ expression in (C.6). The same result should follow from the computation of the partition function on conformally-flat $S^1_\beta \times S^3$ background. The $s > 2$ generalization of (C.1) has the structure

$$Z_{\text{CST},s} = \left[\prod_{k=1}^s \frac{1}{(\det \Delta_{k\perp})^{s-k+1}} \right]^{1/2} = \exp \left[\sum_{m=1}^{\infty} \frac{1}{m} \mathcal{Z}_{\text{CST},s}(q^m) \right], \quad (3.35)$$

where $q = e^{-\beta}$ and $\Delta_{k\perp}$ are defined on 3d tensors appearing in decomposition of φ_s . Using the explicit spectra (and omitting zero modes for fields appearing in transverse decompositions as in [10]) leads to the following one-particle partition function

$$\begin{aligned} \mathcal{Z}_{\text{CST},s}(q) &= \sum_{k=1}^s \sum_{r=1}^{s-k+1} \sum_{n=r-1}^{\infty} 2(n+1)(n+2k+1) q^{n+s-2r+3} \\ &= \frac{2q^2[q^{s+1} - (s+1)^2 q + s(s+2)]}{(1-q)^4} = \frac{2q^2[s(s+2) - q - q^2 - \dots - q^s]}{(1-q)^3}. \end{aligned} \quad (3.36)$$

This is the sum of the contributions of spin $k = 1, \dots, s$ factors in (3.35)

$$\begin{aligned} k = s : & \quad q^{n+s+1}, \quad n \geq 0; \\ k = s-1 : & \quad q^{n+s+1}, \quad n \geq 0, \quad q^{n+s-1}, \quad n \geq 1 \\ k = s-2 : & \quad q^{n+s+1}, \quad n \geq 0, \quad q^{n+s-1}, \quad n \geq 1, \quad q^{n+s-3}, \quad n \geq 2; \quad \text{etc.} \end{aligned} \quad (3.37)$$

The Casimir energy on S^3 can be found the $\beta \rightarrow 0$ expansion of the one-particle partition function as $\mathcal{Z}(e^{-\beta}) = \text{poles} + \text{constant} - 2E\beta + \mathcal{O}(\beta^2)$. From (3.36) we get

$$E_{\text{CST},s} = \frac{1}{720} s(s+1)(6s^3 + 24s^2 + 16s - 13). \quad (3.38)$$

Summing this over s with the cutoff (2.23) gives, in contrast to the CHS case, a non-zero result ($\sum_{s=1}^{\infty} E_{\text{CST},s} = \frac{43}{448}$).

The partition function on S^4 background that generalises the low-spin expressions in (3.19) and (D.5) may be written as (cf. (2.12),(2.32))²⁶

$$Z_{\text{CST},s}(S^4) = \prod_{k=1}^s Z_{k,0}, \quad Z_{k,0} = \left[\frac{\det \Delta_0(M_{0,k}^2)}{\det \Delta_{k\perp}(M_{k,0}^2)} \right]^{1/2}, \quad (3.39)$$

²⁶The "ghost" (numerator) factor here $\prod_{k=1}^s \det \Delta_0(M_{0,k}^2)$, $M_{0,k}^2 = 2 - k - k^2$ is very similar to the one appearing in the determinant of the Weyl-covariant $\nabla^{2s} + \dots$ scalar GJMS operator [53] $\prod_{k=1}^s \det \Delta_0(m_k^2)$, where $m_k^2 = (\frac{1}{2}d - k)(\frac{1}{2}d + k - 1) \Big|_{d=4} = 2 + k - k^2 = M_{0,k-1}^2$.

$$M_{k,0}^2 = 2 + k, \quad M_{0,k}^2 = 2 - k - k^2.$$

The corresponding logarithmic divergence coefficient or conformal anomaly a-coefficient can be found using (3.23)

$$a_{\text{CST},s} = \sum_{k=1}^s \left(a[\Delta_{k\perp}(2+k)] - a[\Delta_0(2-k-k^2)] \right) = \frac{1}{720} s(s+1)^2(3s^2+14s+14). \quad (3.40)$$

Like the sum of Casimir energies (3.38) the sum of conformal anomalies $a_{\text{tot}} = \sum_{s=1}^{\infty} a_{\text{CST},s}$ does not vanish in any natural regularization (in the regularization used in (2.23) one gets $a_{\text{tot}} = -\frac{195}{896}$). There is, of course, no a priori reason why one needs to sum over all ranks s here: in the MHS and CHS cases this summation was implied, in particular, by the AdS/CFT duality and a relation to conserved currents of the boundary theory, but this connection is absent here.

Since the CST field corresponds to the $SO(2,4)$ representation (3.5), it can be associated to a particular field in AdS_5 for the corresponding combination of shadow representations, $(3; \frac{s}{2}, \frac{s}{2}) - (3+s; 0, 0)$. This "maximal depth" conformal field has scalar gauge invariance, i.e. can be interpreted as corresponding to "maximal-depth" partially massless field in AdS_5 [16]. In general, the one-particle partition function of 4d CFT can then be written in terms of AdS_5 partition functions or conformal characters as (see [10, 11])

$$\mathcal{Z}_{\text{CST},s}(q) = \mathcal{Z}_s^-(q) - \mathcal{Z}_s^+(q) = \mathcal{Z}^+(q^{-1}) - \mathcal{Z}^+(q) + \sigma(q), \quad (3.41)$$

$$\mathcal{Z}_s^+(q) = \widehat{\mathcal{Z}}^+(3; \frac{s}{2}, \frac{s}{2}) - \widehat{\mathcal{Z}}^+(s+3; 0, 0) = \frac{(s+1)^2 q^3 - q^{s+3}}{(1-q)^4}. \quad (3.42)$$

$\widehat{\mathcal{Z}}^+(\Delta; j_1, j_2) = (2j_1+1)(2j_2+1) \frac{q^\Delta}{(1-q)^4}$ is the character of a massive (long) conformal group representation $(\Delta; j_1, j_2)$. $\sigma(q)$ is a "correction term" which is expected in the presence of gauge symmetry (scalar gauge invariance here); it is even in $q \rightarrow 1/q$ and absorbs poles in the naive combination $\mathcal{Z}_s^+(q^{-1}) - \mathcal{Z}_s^+(q)$. The minimal choice

$$\sigma(q) = \frac{1}{6} s(s+1)(s+2) + \frac{1}{6} \sum_{n=1}^{s-1} n(n+1)(n+2)(q^{n-s} + q^{s-n}) \quad (3.43)$$

leads to $\mathcal{Z}_{\text{CST},s}(q)$ in (3.36).

In the case of S^4 background the expression for the a-anomaly coefficient can be computed using the AdS_5 relation as in [5, 11], i.e. using (3.27) and generalising (3.28)

$$a_{\text{CST},s} = \widehat{a}(3; \frac{s}{2}, \frac{s}{2}) - \widehat{a}(3+s; 0, 0) = \frac{1}{720} s(s+1)^2(3s^2+14s+14). \quad (3.44)$$

This agrees with the result of the direct computation in $d=4$ in (3.40).

As for the c-anomaly coefficient, its computation directly from the action (3.3) with $s > 2$ on Ricci-flat background is problematic for two reasons: (i) the lack of scalar gauge invariance making the resulting partition function scalar gauge dependent; (ii) the non-minimal nature (lack of factorization) of the corresponding second order differential operator requiring to use more complicated methods (cf. [48]) than the standard algorithm for

the b_4 Seeley coefficient. If one ignores these problems one may formally generalize the known $s = 1$ and $s = 2$ expressions (3.16) to the $s > 2$ Ricci flat case in the way directly analogous to the flat space case (3.32) (cf. also (2.8) and (2.27))

$$\tilde{Z}_{\text{CST},s} = \left[\frac{(\det \Delta_0)^{s+1}}{\det \Delta_{L,s}} \right]^{1/2}. \quad (3.45)$$

Then (3.25) implies that the corresponding $\beta_1 = c - a$ is given by

$$(\tilde{\beta}_1)_{\text{CST},s} = \beta_1[\Delta_{L,s}] - (s+1)\beta_1[\Delta_0] = \frac{1}{720}s(s+1)(3s^4 + 15s^3 + 10s^2 - 30s - 24), \quad (3.46)$$

generalising (3.26).²⁷

We may compare (3.46) with prediction based on dual AdS_5 description. If one uses the expression (3.31) of [50, 51] one gets

$$(\tilde{\beta}_1)'_{\text{CST},s} = \beta'_1(3; \frac{s}{2}, \frac{s}{2}) - \beta'_1(3+s; 0, 0) = \frac{1}{720}s(s+1)(3s^4 + 15s^3 + 10s^2 - 30s - 24), \quad (3.47)$$

i.e. the same result as in (3.46). At the same time, the expression for $\beta_1 = c - a$ for representation $(\Delta; \frac{s}{2}, \frac{s}{2})$ suggested in [11] leads to a different value:

$$(\tilde{\beta}_1)_{\text{CST},s} = (\tilde{\beta}_1)'_{\text{CST},s} - \binom{s+3}{6}, \quad \binom{s+3}{6} = \frac{1}{720}(s-2)(s-1)s(s+1)(s+2)(s+3). \quad (3.48)$$

The two results agree for $s = 0, 1, 2$ but disagree by an integer for $s > 2$.

It is interesting to note that a similar conclusion is reached in the conformal higher spin case. Like (3.45) the factorized expression (2.27) for the CHS partition function on a generic Ricci-flat background is not justified for $s > 2$.²⁸ Still, starting with (2.27) and applying (3.31) to the computation of $(\beta_1)_{\text{CHS},s}$ one gets the same expression as in (2.29); at the same time, $\beta_1(\Delta; \frac{s}{2}, \frac{s}{2})$ or (3.27) suggested in [11] gives the result differing by the same integer as in (3.48).²⁹

$$(\beta_1)_{\text{CHS},s} = (\tilde{\beta}_1)'_{\text{CHS},s} - \binom{s+3}{6}. \quad (3.49)$$

The significance of this observation is not clear at the moment.³⁰

²⁷As in the case of the Casimir energy and a-anomaly, summing (3.47) over s does not give vanishing result in any reasonable regularization.

²⁸One may conjecture that it may still apply to the computation of the β_1 anomaly coefficient. For example, terms that obstruct factorization may depend only on derivatives of the curvature and thus do not contribute to β_1 .

²⁹Note that $\sum_{s=1}^{\infty} \binom{s+3}{6} e^{-\epsilon(s+\frac{1}{2})} \Big|_{\text{fin}} = 0$, so both expressions for β_1 are consistent with the vanishing of the sum of c_s over s (see remark below (2.29)).

³⁰Let us add also that analogous conclusion applies to the scalar GJMS operator discussed in appendix A.1. For example, in $d = 4$ the corresponding value of the a-coefficient that follows from (A.1),(A.17) is $a_r = -\frac{1}{720}r^3(3r^2 - 5)$, in agreement with the AdS_5 expectation (3.27) for the representation $(2+r; 0, 0)$ (the scalar field has dimension $2-r$). The form of this operator on Ricci-flat background is not known explicitly for $r > 4$ (cf. [54–58]). The $r = 3$ operator is singular in $d = 4$ (the singular term is proportional to Bach tensor). Assuming that GJMS operator in $d = 4$ extended to any r (i.e. beyond the "critical" order $r_c = \frac{1}{2}d$) factorises on Ricci-flat background becoming $(\nabla^2)^r$, the corresponding β_1 coefficient is readily found to be r times the standard scalar one, i.e. $\beta'_1 = \frac{1}{180}r$. This is the same result that follows from (3.31) for the representation $(2+r; 0, 0)$. At the same time, from (3.31) we get $\beta_1 = \frac{1}{720}r(-3r^4 + 15r^2 - 8)$. Thus here $\beta_1 = \beta'_1 - \frac{1}{2}\binom{r+2}{5}$, which is similar to (3.48) and (3.49).

Acknowledgments

We thank M. Grigoriev, E. Joung, R. Metsaev and D. Toms for useful discussions. This work is a part of collaboration supported by the Russian Science Foundation grant 14-42-00047 and associated with Lebedev Institute. The work of AAT was supported also by the ERC Advanced grant No.290456. AAT would like also to thank the Galileo Galilei Institute for Theoretical Physics for the hospitality and the INFN for partial support while part of this work was in progress.

A Relation between partition functions on S^4 and on AdS_5

The aim of this Appendix is to consider in detail the relation (2.30) by computing the determinants involved. As discussed in sections 2.1.3, 2.2.3 one should carefully correlate the UV regularization in 4d and IR regularization in 5d. We shall follow the approach of [14] where dimensional regularisation $4 \rightarrow d = 4 - \varepsilon$ was used in demonstrating a similar relation in spin 0 case. $\varepsilon \rightarrow 0$ plays the role of a common IR/UV regulator on the AdS/CFT sides. This allows a careful separation between the divergent pole terms $\sim \frac{1}{\varepsilon}$ and finite remainder, accounting, in particular, for the IR finite terms in the AdS_5 partition function ignored in direct zeta-function regularization in [8]. Matching the finite terms turns out to be quite subtle. We will first illustrate this on the example of the partition function of GJMS operators on S^4 extending the analysis of [14, 30]. We shall then turn to the more complicated but structurally similar case of the CHS fields on S^4 related to MHS fields on AdS_5 .

A.1 Matching partition function of GJMS operators on S^4 and ratio of scalar partition functions on AdS_5

GJMS operators [53] are the unique Weyl-covariant $\nabla^{2r} + \dots$ operators in d dimensions defined on scalars. In $d = 4$ the $r = 1$ case is familiar $-\nabla^2 + \frac{1}{6}R$ and the $r = 2$ operator $\nabla^4 + \dots$ was constructed in [59, 60]. Their general properties are discussed in [54–58]. On an Einstein space background, they factorize into a product of r scalar 2nd-order operators

$$\mathcal{D}_{(2r)} = \prod_{k=1}^r (-\nabla^2 + q_k R), \quad q_k = \frac{(\frac{d}{2} - k)(\frac{d}{2} + k - 1)}{d(d-1)}. \quad (\text{A.1})$$

Using that on a unit sphere S^d (with $R = d(d-1)$) the eigenvalues and multiplicities of $-\nabla^2 + M^2$ are

$$\lambda_n = n(n+d-1) + M^2, \quad d_n = \frac{(2n+d-1)\Gamma(n+d-1)}{\Gamma(d)\Gamma(n+1)}, \quad (\text{A.2})$$

we get explicitly³¹

$$\ln \det \mathcal{D}_{(2r)} = \sum_{n=0}^{\infty} \frac{(d+2n-1)\Gamma(d+n-1)}{\Gamma(d)\Gamma(n+1)} \ln \left[\prod_{k=1}^r (n+k+\frac{d}{2}-1)(n-k+\frac{d}{2}) \right]. \quad (\text{A.3})$$

³¹As discussed in [14], in dimensional regularisation one has a useful relation $\sum_{n=0}^{\infty} d_n = 0$ allowing to drop constants under the logarithm of eigenvalues.

From (A.3), it is clear that the critical order $r = r_c \equiv \frac{d}{2}$ is special: (i) for $r < r_c$ all eigenvalues are positive; (ii) for $r = r_c$ there is one zero mode; (iii) for $r > r_c$ there are zero and negative eigenvalues (see also [61]).

GJMS theory may be viewed as an induced conformal theory on S^d boundary of AdS_{d+1} corresponding to a standard 2nd-derivative scalar in AdS_{d+1} and the analogue of (2.30) reads³²

$$Z_{(2r)}(S^d) = \frac{Z_0^-(AdS_{d+1})}{Z_0^+(AdS_{d+1})}, \quad (A.4)$$

$$Z_{(2r)} \equiv (\det \mathcal{D}_{(2r)})^{-1/2}, \quad Z_0 = (\det \hat{\Delta}_0)^{-1/2}, \quad \hat{\Delta}_0 = -\nabla^2 + m^2, \quad (A.5)$$

where Z_0^\pm is the partition function of a massive scalar operator in AdS_{d+1} with $m^2 = \Delta(\Delta - d) = r^2 - \frac{d^2}{4}$, i.e. with the associated operators having dimensions $\Delta_+ = \frac{d}{2} + r$ and $\Delta_- = \frac{d}{2} - r$.³³

We may now follow the procedure in [14] to evaluate (A.3) by first using that

$$\prod_{k=1}^r (n+k+\frac{d}{2}-1)(n-k+\frac{d}{2}) = \frac{\Gamma(n+\frac{d}{2}+r)}{\Gamma(n+\frac{d}{2}-r)}, \quad (A.6)$$

and also replacing r by $\Delta \equiv \Delta_+ = \frac{d}{2} + r$ (we shall use the notation $\mathcal{D}_{(2r)} \rightarrow \mathcal{D}(\Delta)$). Formally treating Δ as a continuous variable, (A.3) becomes

$$\ln \det \mathcal{D}(\Delta) = \sum_{n=0}^{\infty} \frac{(d+2n-1)\Gamma(d+n-1)}{\Gamma(d)\Gamma(n+1)} \ln \frac{\Gamma(n+\Delta)}{\Gamma(n+d-\Delta)}. \quad (A.7)$$

It is convenient to first take derivative of (A.7) with respect to Δ , do the sum and then integrate over Δ , fixing the integration constant by demanding that the result should vanish at $r = 0$ or $\Delta = \frac{d}{2}$ when the GJMS operator becomes trivial. The sum

$$\frac{\partial}{\partial \Delta} \ln \det \mathcal{D}(\Delta) = \sum_{n=0}^{\infty} \frac{(d+2n-1)\Gamma(d+n-1)}{\Gamma(d)\Gamma(n+1)} [\psi(n+\Delta) + \psi(n+d-\Delta)]. \quad (A.8)$$

was already computed in [14]:

$$\frac{\partial}{\partial \Delta} \ln \det \mathcal{D}(\Delta) = \frac{\Gamma(-\frac{d}{2})\Gamma(d-1)(d-2\Delta)\Gamma(\Delta)\Gamma(d-\Delta) \sin[\frac{\pi}{2}(d-2\Delta)]}{2^d \sqrt{\pi} \Gamma(d-1)\Gamma(\frac{d+1}{2})}. \quad (A.9)$$

³²In this Appendix we add hats to Laplacian operators not to confuse them with dimension parameter Δ .

³³Let us note that [14] considered the case of a generic massive AdS_{d+1} scalar with non-integer $r = \sqrt{m^2 + \frac{d^2}{4}}$ when the associated induced boundary theory is non-local: the kinetic operator is inverse of $K(x, x') = \langle J(x)J(x') \rangle = [s(x, x')]^{-2\Delta_-}$ where $s(x, x')$ is the geodesic distance on (conformally) flat space. Then the l.h.s. of (A.4) is replaced by the partition function of the corresponding non-local operator which may be interpreted in terms of a double trace deformation of CFT corresponding to the change of boundary conditions for the dual AdS_{d+1} theory [62, 63]. For an integer r the non-local kinetic operator $[s(x, x')]^{-2\Delta_+}$ has the leading singular being the local GJMS operator acting on a delta-function (see also [30, 7]); thus the inverse of the determinant of $K(x, x')$ is effectively replaced by the determinant of the local conformally-covariant GJMS operator as in (A.4).

Let us now turn to the AdS_{d+1} side. Here one can express the derivative of the r.h.s. of (A.4) over the scalar mass or Δ in terms of an integral of the trace of the corresponding difference of the AdS_{d+1} scalar bulk-to-bulk propagators [14]. In general, for any spin $s \geq 0$ one can utilise the expression in eq.(71) of [64] for the mass² derivative of the difference of the logarithms of the partition functions (or, equivalently, the difference of the bulk-to-bulk propagators) for the spin s symmetric transverse traceless field in AdS_{d+1} with the kinetic operator (cf. (2.12)) $\hat{\Delta}_{s\perp} = -\nabla^2 + m^2$, $m^2 = \Delta(\Delta - d) - s$ corresponding to the standard (+) and alternative (-) boundary conditions (or dimensions $\Delta_+ = \Delta$ and $\Delta_- = d - \Delta$)³⁴

$$\begin{aligned} \frac{\partial}{\partial \Delta} \ln \frac{\det_- \hat{\Delta}_{s\perp}}{\det_+ \hat{\Delta}_{s\perp}} &= \pi^{\frac{d}{2}} \Gamma(-\frac{d}{2}) \frac{(2s + d - 2)(\Delta - \frac{d}{2}) \Gamma(s + d - 2)}{\Gamma(d - 1) \Gamma(s + 1)} \\ &\times \frac{(\Delta + s - 1)(\Delta - s - d + 1) \Gamma(\Delta - 1) \Gamma(d - 1 - \Delta) \sin[\frac{\pi}{2}(d - 2\Delta)]}{2^{d-1} \pi^{\frac{d+1}{2}} \Gamma(\frac{d+1}{2})}. \end{aligned} \quad (\text{A.10})$$

In (A.10) we separated the factor of the dimensionally regularised volume of AdS_{d+1} (cf. (2.15))

$$V_{d+1} = \pi^{\frac{d}{2}} \Gamma(-\frac{d}{2}). \quad (\text{A.11})$$

Comparing (A.9) with the $s = 0$ case of (A.10), we find

$$\frac{\partial}{\partial \Delta} \ln \det \mathcal{D}(\Delta) = \frac{\partial}{\partial \Delta} \ln \frac{\det_- \hat{\Delta}_0}{\det_+ \hat{\Delta}_0}, \quad (\text{A.12})$$

which is indeed in agreement with (A.4).

Expanding around $d = 4$, i.e. setting $d = 4 - \varepsilon$ with $\varepsilon \rightarrow 0$, one can check that the relation (A.12) holds for both the pole term and the finite remainder. The expansion of (A.9) is

$$\begin{aligned} \frac{\partial}{\partial \Delta} \ln \det \mathcal{D}(\Delta) &= \frac{1}{\varepsilon} P(\Delta) + F(\Delta) + \mathcal{O}(\varepsilon), \\ P(\Delta) &= -\frac{1}{6}(\Delta - 3)(\Delta - 2)^2(\Delta - 1), \\ F(\Delta) &= \frac{1}{72}(\Delta - 2)(\Delta - 1) \left[12(\Delta - 3)(\Delta - 2) \gamma_E + (107 - 25\Delta) \Delta \right. \\ &\quad \left. - 6\pi(\Delta - 3)(\Delta - 2) \cot(\pi\Delta) + 12(\Delta - 3)(\Delta - 2) \psi^{(0)}(3 - \Delta) - 108 \right]. \end{aligned} \quad (\text{A.13})$$

Let us now integrate (A.12) in the interval $\frac{d}{2} \leq \Delta \leq \frac{d}{2} + r$. The integral of the pole is

$$\frac{1}{\varepsilon} \int_{\frac{d}{2}}^{\frac{d}{2}+r} d\Delta P(\Delta) = -\frac{1}{90} r^3 (3r^2 - 5) \frac{1}{\varepsilon} + \frac{1}{12} r^2 (r^2 - 1) + \mathcal{O}(\varepsilon). \quad (\text{A.14})$$

Here the coefficient of the singular part is in agreement (after taking into account normalizations) with the conformal anomaly coefficient $a_r = -\frac{1}{720} r^3 (3r^2 - 5)$ for the GJMS field that can be computed directly from the Seeley coefficients B_4 for the operators in (A.1).

³⁴More precisely, eq.(71) of [64] gives $\frac{1}{2} \frac{\partial}{\partial m^2} \ln \frac{\det_- \hat{\Delta}_{s\perp}}{\det_+ \hat{\Delta}_{s\perp}} = [4(\Delta - \frac{d}{2})]^{-1} \frac{\partial}{\partial \Delta} \ln \frac{\det_- \hat{\Delta}_{s\perp}}{\det_+ \hat{\Delta}_{s\perp}}$. Note that here $m^2 = M^2 \sigma = -M^2$ in the notation in (2.10),(2.12).

The full finite part of $\ln \det \mathcal{D}(\Delta)$ is the sum of the second term in (A.14) and the integral of the second term $F(\Delta)$ in (A.13) (here we may set $d = 4$, i.e. $\Delta = 2 + r$)

$$X(\Delta) = \int_2^\Delta d\Delta' F(\Delta'), \quad \Delta = 2 + r. \quad (\text{A.15})$$

The function $X(\Delta)$ has poles at $\Delta = 4, 5, 6, \dots$. These poles are associated with the previously mentioned zero eigenvalues appearing in (A.3) for $r \geq r_c = \frac{d}{2}$.³⁵ The poles of $X(\Delta)$ inside the interval $(2, 2 + r)$ can be evaluated by taking the principal part of the integral. The rightmost pole will give a singular term $\sim \ln [\Delta - (2 + r)]$. As a result, we find for $d \rightarrow 4, \Delta \rightarrow 2 + r$:

$$\ln \det \mathcal{D}(\Delta) = \left(\frac{1}{\varepsilon} - \gamma_E \right) P(\Delta) + F(\Delta) + \mathcal{O}(\varepsilon), \quad (\text{A.16})$$

$$P(2 + r) = 8a_r = -\frac{1}{90}r^3(3r^2 - 5), \quad (\text{A.17})$$

$$\begin{aligned} F(2 + r - \delta) \Big|_{\delta \rightarrow 0} &= \frac{1}{3} \ln A - \frac{2}{3} \zeta'(-3) + \frac{1}{6} (r^2 - 1) r^2 \ln \Gamma(r) \\ &+ \frac{1}{3} (r - 2r^3) \psi^{(-2)}(1 - r) - \frac{1}{4\pi^2} (r^2 - 1) \zeta(3) + \left(\frac{1}{3} - 2r^2 \right) \psi^{(-3)}(1 - r) \\ &+ \frac{1}{12} (r^2 - r^4) \ln(2\pi) - 4r \psi^{(-4)}(1 - r) - 4\psi^{(-5)}(1 - r) \\ &+ \frac{1}{2160} (-150r^5 + 45r^4 + 130r^3 - 90r^2 - 22) + \frac{1}{12} (r^2 - 1) r^2 \ln \delta + \mathcal{O}(\delta), \end{aligned} \quad (\text{A.18})$$

where A is the Glaisher constant, i.e. $\ln A = -\frac{1}{2\pi^2} \zeta'(2) + \frac{1}{12} \ln(2\pi) + \frac{1}{12} \gamma_E$. As anticipated, the last term in (A.18) is due to the zero modes appearing in the original expression (A.3) when $r \geq r_c = 2$ in $d = 4$. The prefactor of $\ln \delta$ is indeed the sum of the associated multiplicities. When the zero modes are projected out and thus $\ln \delta$ terms are omitted, (A.18) is given by a finite expression containing various transcendental constants and logarithms of integers. The expressions for the polylogarithms in (A.18) can be put in a more explicit form so that we find after dropping the $\ln \delta$ terms

$$F(2 + r) = \frac{1}{3} r (2r^2 - 1) \ln A - \frac{2}{3} r \zeta'(-3) - \frac{1}{720} r (60r^2 - 31) + \mathcal{L}_r, \quad (\text{A.19})$$

where \mathcal{L}_r is a sum of logarithms of integers which general dependence on r we did not find.³⁶

A.2 Matching CHS partition function on S^4 and ratio of MHS partition functions on AdS_5

Let us now repeat the above discussion of the scalar case for the $s > 0$ CHS theory on S^4 to demonstrate the relation (2.30) to MHS partition function on AdS_5 . Once again, one

³⁵In addition, $X(\Delta)$ contains imaginary terms (multiples of π) that are related to negative eigenvalues in (A.3) appearing for $r > r_c$ (note also that for $r \geq r_c = \frac{d}{2}$ one has $\Delta_- = \frac{d}{2} - r < 0$). In the following, we shall formally omit these imaginary contributions that appear on both sides of (A.4).

³⁶Explicitly, for low values of $r = 1, \dots, 5$ we get $\mathcal{L}_{1,2,3} = 0$, $\mathcal{L}_4 = 2 \ln 2$, $\mathcal{L}_5 = 2 \ln 3 + 12 \ln 2$, i.e.

$$\begin{aligned} F(3) &= \frac{\ln A}{3} - \frac{2}{3} \zeta'(-3) - \frac{29}{720}, \quad F(4) = \frac{14 \ln A}{3} - \frac{4}{3} \zeta'(-3) - \frac{209}{360}, \quad F(5) = 17 \ln A - 2 \zeta'(-3) - \frac{509}{240}, \\ F(6) &= \frac{124 \ln A}{3} - \frac{8}{3} \zeta'(-3) - \frac{929}{180} + 2 \ln 2, \quad F(7) = \frac{245 \ln A}{3} - \frac{10}{3} \zeta'(-3) - \frac{1469}{144} + 2 \ln 3 + 12 \ln 2. \end{aligned}$$

should use the dimensional regularization that provides UV regularization on the CHS boundary side and IR regularization on the MHS bulk side at the same time. The matching of the logarithmically divergent parts was already demonstrated in [5, 7]. The use of dimensional regularization is essential on AdS_5 side as otherwise one misses the crucial finite part of $\ln Z_{\text{MHS}}$ that is always present in $\ln Z_{\text{CHS}}$. Below we shall limit our analysis to matching the most transcendental parts of the finite contributions to the partition functions on the two sides of (2.30), i.e. the terms proportional to $\ln A$ and $\zeta'(-3)$ as in (A.19).

The starting point on the CHS side should be the $d = 4 \rightarrow d = 4 - \varepsilon$ generalization (cf. (2.24)) of the partition function (2.32) on S^4 . As we will be interested only in the most transcendental finite part of

$$G_s \equiv -\ln Z_{\text{CHS},s}(S^4), \quad (\text{A.20})$$

we may ignore the analog of the $d - 4$ power factor in (2.24) (it will contribute only to rational finite terms) and start directly with (2.32) in $d = 4$ introducing some fiducial UV cutoff (the most transcendental terms in (A.20) should not depend on its choice, cf. [30]).

Using the known spectra of $\Delta_{s\perp}$ operators on S^4 (see, e.g., [65]) we find from (2.32)

$$G_s = \sum_n^{\Lambda-s-2} \sum_{k=0}^{s-1} Q_s(n, k), \quad \Lambda \rightarrow \infty \quad (\text{A.21})$$

$$Q_s(n, k) = \frac{1}{12}(n+1)(2s+1)(n+2s+2)(2n+2s+3) \ln[(n-k+s+1)(n+k+s+2)] \\ - \frac{1}{12}(n+1)(2k+1)(n+2k+2)(2n+2k+3) \ln[(n+k-s+1)(n+k+s+2)]$$

Here we introduced a sharp UV cutoff N for the sum over n and did not specify the starting values of n in the sum, which should be different for various terms being related to projection of zero modes: these shifts will not be relevant for the calculation of the most transcendental terms in G_s . We made a particular choice of the upper limit as $\Lambda - s - 2$ to reproduce the already known (from proper time cutoff computation [7]) value of the a -coefficient (2.33) of the logarithmic divergence.

For example, in the $s = 1$ case we obtain from (A.21)

$$G_1 = \frac{1}{24}\Lambda^4(4\ln\Lambda - 1) + \frac{1}{3}\Lambda^3\ln\Lambda - \frac{1}{36}\Lambda^2(60\ln\Lambda - 17) + \frac{1}{6}\Lambda(\ln\Lambda - 1) + \frac{31}{45}\ln\Lambda \\ - \frac{11}{3}\ln A - \frac{2}{3}\zeta'(-3) - \frac{52}{135} - \frac{9}{2}\ln 3 - 5\ln 2 + \mathcal{O}(\Lambda^{-1}), \quad (\text{A.22})$$

where A is Glaisher constant. G_s turns out to have a similar structure also for $s > 1$, i.e. one finds

$$G_s = A_s(\Lambda) + 4a_s \ln \Lambda + F_s + \mathcal{O}(\Lambda^{-1}), \quad a_s = \frac{1}{720}s^2(s+1)^2(14s^2 + 14s + 3), \\ F_s = -\frac{1}{6}s(s+1)(5s^2 + 5s + 1) \ln A - \frac{1}{3}s(s+1)\zeta'(-3) + \dots, \quad (\text{A.23})$$

where $A_s(\Lambda)$ denote all divergent terms with positive powers of Λ and a_s is the same conformal anomaly coefficient as in (2.33)³⁷ and dots stand for rational numbers plus logarithms of integers.

³⁷To recall normalizations (see, e.g., [65]), $\ln Z = B_4 \ln L + \dots$ where L is a UV cutoff and $B_4 = \frac{1}{(4\pi)^2} \int d^4x \sqrt{g} b_4 \Big|_{S^4} = -4a$, where as in (3.22) we have $b_4 = -aR^*R^* + cC^2$.

Let us now turn to the AdS_5 side. Each of the MHS partition functions in the r.h.s. of (2.30) is given by the ratio of the physical and ghost determinants as in (2.12) with the operators $\hat{\Delta}_{s\perp}$ and $\hat{\Delta}_{s-1\perp}$ having the "mass" terms $m^2 = \Delta(\Delta - d) - s$ corresponding to dimensions $\Delta \equiv \Delta_+ = d + s - 2$ and $d + s - 1$ respectively. Using the key relation (A.10) and integrating over Δ we can first find the pole parts in the limit $d = 4 - \varepsilon \rightarrow 4$:³⁸

$$\ln \frac{\det_- \hat{\Delta}_{s\perp}}{\det_+ \hat{\Delta}_{s\perp}} \Big|_{\Delta=d+s-2} = \frac{1}{90} s^3 (s+1)^2 (2s^2 + 10s + 5) \frac{1}{\varepsilon} + \dots, \quad (\text{A.24})$$

$$\ln \frac{\det_- \hat{\Delta}_{s-1\perp}}{\det_+ \hat{\Delta}_{s-1\perp}} \Big|_{\Delta=d+s-1} = \frac{1}{90} s^2 (s+1)^3 (2s^2 - 6s - 3) \frac{1}{\varepsilon} + \dots \quad (\text{A.25})$$

Defining

$$G_s \equiv -\ln \frac{Z_{\text{MHS},s}^-}{Z_{\text{MHS},s}^+} = \frac{1}{2} \left(\ln \frac{\det_- \hat{\Delta}_{s\perp}}{\det_+ \hat{\Delta}_{s\perp}} \Big|_{\Delta=d+s-2} - \ln \frac{\det_- \hat{\Delta}_{s-1\perp}}{\det_+ \hat{\Delta}_{s-1\perp}} \Big|_{\Delta=d+s-1} \right) \quad (\text{A.26})$$

and comparing to (A.20), (A.23) we check that the logarithmically divergent terms in the l.h.s. and r.h.s. parts of (2.30) match (cf. (A.12))

$$(G_s)_{\ln \Lambda} = (G_s)_{\frac{1}{\varepsilon}}. \quad (\text{A.27})$$

The finite part of G_s can be computed by integrating the finite part of (A.10), separately for the physical and the ghost contributions. As in the spin zero case discussed in the previous subsection, there are poles along the integration interval that should be related to the zero modes on the CHS side, i.e. to the zero eigenvalue contributions in (A.21) that should be projected out. The treatment of these poles is completely analogous to the one in the scalar case discussed below (A.15). We find

$$\begin{aligned} & \frac{1}{2} \left(\ln \frac{\det_- \hat{\Delta}_{s\perp}}{\det_+ \hat{\Delta}_{s\perp}} \Big|_{\Delta=d+s-2-\delta} - \ln \frac{\det_- \hat{\Delta}_{s-1\perp}}{\det_+ \hat{\Delta}_{s-1\perp}} \Big|_{\Delta=d+s-1-\delta} \right)_{\text{fin}} \\ &= -\frac{1}{6} (4s^3 + 6s^2 - 1) \ln A - \frac{1}{12} s^2 (s+1)^2 (2s+1) \ln [\Gamma(s)\Gamma(s+1)] \\ & \quad - \frac{1}{90} s^2 (s+1)^2 (14s^2 + 14s + 3) \gamma_E + \frac{1}{12} s (2s^4 + 5s^3 + 2s^2 - 2s - 1) \ln(2\pi) \\ & \quad - \frac{1}{3} (2s+1) \zeta'(-3) + \frac{1}{8\pi^2} (2s+1) \zeta(3) + 2s^2 \psi^{(-5)}(-s) + 2s^2 (s+1) \psi^{(-4)}(-s) \\ & \quad + \frac{1}{6} (5s^2 + 12s + 6) s^2 \psi^{(-3)}(-s) + \frac{1}{180} (s+1)^3 (2s^2 - 6s - 3) s^2 \psi^{(0)}(s+1) \\ & \quad - \frac{1}{6} (5s^4 + 8s^3 - 4s - 1) \psi^{(-3)}(1-s) - \frac{1}{180} (s+1)^2 s^3 (2s^2 + 10s + 5) \psi^{(0)}(s+2) \\ & \quad + \frac{1}{6} (s^3 + 5s^2 + 6s + 2) s^2 \psi^{(-2)}(-s) + \frac{1}{6} (-s^5 + 4s^3 + 4s^2 + s) \psi^{(-2)}(1-s) \\ & \quad - 2s(s+1)^2 \psi^{(-4)}(1-s) - 2(s+1)^2 \psi^{(-5)}(1-s) \\ & \quad + \frac{1}{4320} (1204s^6 + 2478s^5 + 1035s^4 - 620s^3 - 417s^2 - 80s - 22) - \frac{1}{12} s^2 (s+1)^2 (2s+1) \ln \delta \end{aligned} \quad (\text{A.28})$$

One can check that the coefficient of $\ln \delta$ term is indeed the sum of multiplicities of the zero eigenvalues.

³⁸An alternative computation of these IR singular terms in these AdS_5 partition functions was first done in [5].

Dropping $\ln \delta$ -terms, i.e. concentrating on the remaining finite contribution analogous to (A.19), its most transcendental part can be put into the following simple form³⁹

$$(G_s)_{\text{fin}} = -\frac{1}{6}q_s \ln A - \frac{1}{3}\nu_s \zeta'(-3) + \dots, \quad (\text{A.29})$$

$$q_s = s(s+1)(5s^2 + 5s + 1) = \nu_s(5\nu_s + 1), \quad \nu_s = s(s+1), \quad (\text{A.30})$$

where dots stand again for a rational contribution plus a string of logarithms of primes plus a γ_E term that may be combined with the $\frac{1}{\epsilon}$ pole term as in (A.16). Comparing with (A.23), we conclude that the most transcendental terms in finite parts of G_s (A.20) and G_s (A.26) match, i.e. in addition to (A.27) we get

$$(G_s)_{\ln A, \zeta'(-3)} = (G_s)_{\ln A, \zeta'(-3)}. \quad (\text{A.31})$$

This provides a non-trivial confirmation of the relation (2.30).

Finally, let us sum over all spins to provide a check of the $(Z_{\text{CHS}})_{\text{tot}} = 1$ relation (2.31) using the same summation prescription (2.23) that implies the vanishing of the total number of dynamical degrees of freedom of the CHS theory and the total value of the conformal anomaly coefficient a in (2.33), (A.23) [5, 8]

$$\nu_{\text{tot}} = \sum_{s=0}^{\infty} \nu_s e^{-\epsilon(s+\frac{1}{2})} \Big|_{\text{fin.}} = 0, \quad a_{\text{tot}} = \sum_{s=0}^{\infty} a_s e^{-\epsilon(s+\frac{1}{2})} \Big|_{\text{fin.}} = 0. \quad (\text{A.32})$$

We observe that the same is true also for the sum of q_s coefficients in (A.30)⁴⁰

$$q_{\text{tot}} = \sum_{s=0}^{\infty} q_s e^{-\epsilon(s+\frac{1}{2})} \Big|_{\text{fin.}} = 0. \quad (\text{A.33})$$

This implies the vanishing not only of the UV singular part but also of the most transcendental finite part of $\ln(Z_{\text{CHS}})_{\text{tot}}$, i.e. of (A.29) summed over all spins.

B Conditions for scalar gauge invariance of conformal symmetric rank 2 tensor in curved background

Let us consider the variation of the Lagrangian (3.6) under the transformation (3.12). Integrating by parts in the linear in σ terms (moving covariant derivatives from σ to the background and $\varphi_{\mu\nu}$) the condition for invariance may be written as

$$\begin{aligned} & -2\nabla^\mu R \nabla_\nu \varphi_\mu{}^\nu - R \nabla_\nu \nabla_\mu \varphi^{\mu\nu} - \varphi^{\mu\nu} \nabla_\nu \nabla_\mu R - 6\nabla_\nu \nabla_\mu \nabla_\rho \nabla^\rho \varphi^{\mu\nu} + 12\varphi^{\mu\nu} \nabla_\nu \nabla_\rho R_\mu{}^\rho \\ & + 8\nabla_\nu \nabla_\rho \nabla^\rho \nabla_\mu \varphi^{\mu\nu} + 12\nabla_\mu \varphi^{\mu\nu} \nabla_\rho R_\nu{}^\rho + 12R^{\mu\nu} \nabla_\rho \nabla_\nu \varphi_\mu{}^\rho - 3R^{\mu\nu} \nabla_\rho \nabla^\rho \varphi_{\mu\nu} \\ & - 3\varphi^{\mu\nu} \nabla_\rho \nabla^\rho R_{\mu\nu} - 2\nabla_\rho \nabla^\rho \nabla_\nu \nabla_\mu \varphi^{\mu\nu} + 12\nabla_\nu R_{\mu\rho} \nabla^\rho \varphi^{\mu\nu} - 6\nabla_\rho R_{\mu\nu} \nabla^\rho \varphi^{\mu\nu} \\ & + 6\omega(\varphi^{\mu\nu} \nabla_\lambda \nabla_\rho C_\mu{}^\rho{}_\nu{}^\lambda + C_{\mu\rho\nu\lambda} \nabla^\lambda \nabla^\rho \varphi^{\mu\nu} + 2\nabla_\lambda C_{\mu\rho\nu}{}^\lambda \nabla^\rho \varphi^{\mu\nu}) = 0. \end{aligned} \quad (\text{B.1})$$

³⁹Notice that like in (A.19) the coefficient of $\zeta'(-3)$ happens to be proportional to the number of dynamical degrees of freedom.

⁴⁰Explicitly, $\sum_{s=0}^{\infty} s(s+1)(5s^2 + 5s + 1) e^{-\epsilon(s+\frac{1}{2})} = \frac{120}{\epsilon^5} - \frac{3}{\epsilon^3} + \frac{1}{16\epsilon} + \mathcal{O}(\epsilon)$. More generally, $\sum_{s=0}^{\infty} (\nu_s)^n e^{-\epsilon(s+\frac{1}{2})} \Big|_{\text{fin.}} = 0$ for $\nu_s = s(s+1)$ and any integer n .

Commuting the covariant derivatives in the $\nabla^4\varphi$ terms we find that they cancel against the $\nabla^3\varphi$ terms. The remaining $\nabla^2\varphi$ terms take the form

$$\frac{1}{3}R\nabla_\nu\nabla_\mu\varphi^{\mu\nu} - \frac{4}{3}R^{\mu\nu}\nabla_\nu\nabla_\rho\varphi_\mu^\rho + R^{\mu\nu}\nabla_\rho\nabla^\rho\varphi_{\mu\nu} + 2(2+\omega)C_{\mu\rho\nu\lambda}\nabla^\lambda\nabla^\rho\varphi^{\mu\nu}. \quad (\text{B.2})$$

Since $\varphi_{\mu\nu}$ is symmetric traceless this gives the condition $\tilde{K}_{\mu\nu\rho\sigma} = K_{\mu\nu\rho\lambda} - \frac{1}{4}g_{\mu\nu}K^\sigma_{\sigma\rho\lambda} = 0$, where

$$K_{\mu\nu\rho\lambda} = g_{\rho\lambda}R_{\mu\nu} - \frac{2}{3}(g_{\nu\rho}R_{\mu\lambda} + g_{\mu\rho}R_{\nu\lambda}) + \frac{1}{6}(g_{\mu\rho}g_{\nu\lambda} + g_{\mu\lambda}g_{\nu\rho})R + (2+\omega)(C_{\mu\rho\nu\lambda} + C_{\nu\rho\mu\lambda}) = 0. \quad (\text{B.3})$$

Then the contraction $g^{\mu\rho}\tilde{K}_{\mu\nu\rho\sigma} = 0$ gives the requirement that the background should be Einstein

$$R_{\mu\nu} = \frac{1}{4}Rg_{\mu\nu}. \quad (\text{B.4})$$

Using this in (B.3) and $\tilde{K}_{\mu\nu\rho\sigma} = 0$ gives further constraint

$$(2+\omega)C_{\rho(\mu\nu)\lambda} = 0. \quad (\text{B.5})$$

If $2+\omega \neq 0$ then $C_{\rho(\mu\nu)\lambda} = 0$ combined with the first Bianchi identity $C_{\mu[\nu\rho\lambda]} = C_{\mu\nu\rho\lambda} + C_{\mu\rho\lambda\nu} + C_{\mu\lambda\nu\rho} = 0$, implies that $C_{\mu\nu\rho\lambda} = 0$, i.e. the space should be conformally flat. The alternative is to assume that

$$\omega = -2. \quad (\text{B.6})$$

Then the remaining part of the variation (B.1) gives the condition $\nabla_\lambda C^\lambda_{(\mu\nu)\rho} = 0$, i.e. $\nabla_\lambda C^\lambda_{\mu\nu\rho} = 0$. This is automatically satisfied as a consequence of the Einstein condition (B.4) and the second Bianchi identity $\nabla_{[\lambda}R_{\mu\nu]\sigma\rho} = 0$.

In conclusion, the Lagrangian (3.6) admits the invariance (3.12) if the background is Einstein and is also conformally flat or it is generic but then ω is to be fixed as in (B.6).

A similar analysis for $s > 2$ implies that imposing the Einstein condition (B.4) is not enough to ensure the scalar gauge invariance of (3.3) for any value of ω unless the space is also conformally flat.

C Partition function of conformal symmetric rank 2 tensor on $S^1 \times S^3$

Starting with the rank 2 tensor Lagrangian (3.6) on conformally-flat $S^1_\beta \times S^3$ space and performing 1+3 decomposition $\varphi_{\mu\nu} = (\varphi_{ij}, \varphi_{0i}, \varphi_{00})$, $\varphi_{ij} = \varphi_{ij}^\perp + \nabla_{(i}V_{j)}^\perp + \dots$ one can represent the resulting partition function as (cf. (3.16))

$$Z_{\text{CST},2} = \left[\frac{1}{\det \Delta_{2\perp} \det' \Delta_{1\perp} \det \Delta_{1\perp}} \right]^{1/2}, \quad (\text{C.1})$$

where Δ_n operators act on 3d n -tensors. We consider unit-radius S^3 and S^1 of length β . Here $\Delta_{2\perp} = -\nabla^2 + 3 = -\partial_0^2 - \nabla^2 + 3$ and the two vector operators acting on V_i^\perp and φ_{0i} have similar form as in Maxwell theory (cf. [10]). The spectrum of the rank 2 operator is found to be

$$\lambda_{k,n} = \left(\frac{2\pi k}{\beta}\right)^2 + w_n^2, \quad w_n^2 = (n+2)(n+4) - 2 + 3 = (n+3)^2. \quad (\text{C.2})$$

As a result, $\ln Z_{\text{CST},2} = \sum_{m=1}^{\infty} \frac{1}{m} \mathcal{Z}(q^m)$, where $q = e^{-\beta}$ and the one-particle partition function $\mathcal{Z}(q)$ is

$$\begin{aligned} \mathcal{Z}(q) &= \underbrace{\sum_{n=0}^{\infty} 2(n+1)(n+5)q^{n+3}}_{\varphi_{ij}^{\perp}} + \underbrace{\sum_{n=1}^{\infty} 2(n+1)(n+3)q^{n+1}}_{V_i^{\perp}} + \underbrace{\sum_{n=0}^{\infty} 2(n+1)(n+3)q^{n+3}}_{\varphi_{0i}} \\ &= \frac{2(8q^2 - 9q^3 + q^5)}{(1-q)^4}. \end{aligned} \quad (\text{C.3})$$

Here the V_i^{\perp} contribution starts at $n = 1$ because, as in [10], the 6 zero modes of this vector drop out.

The same expression can be found by counting the conformal operators in flat space \mathbb{R}^4 (cf. [66, 10]). The flat-space equations (3.8) may be written in terms of a field strength invariant under the scalar gauge transformations [67]

$$H_{\mu\nu\rho} = \partial_{[\mu} \varphi_{\nu]\rho} - \frac{1}{3} \delta_{\rho[\mu} \partial_{\alpha} \varphi_{\nu]}^{\alpha}, \quad H_{\mu\nu}{}^{\nu} = 0, \quad \epsilon^{\alpha\mu\nu\rho} H_{\mu\nu\rho} = 0. \quad (\text{C.4})$$

The number of independent components of dimension 2 field $H_{\mu\nu\rho}$ is $6 \times 4 - 4 - 4 = 16$. The equations of motion together with "Bianchi" identities then take the form

$$\partial^{\mu} H_{\mu(v\rho)} = 0, \quad \partial^{\mu} \tilde{H}_{\mu(v\rho)} = 0, \quad \tilde{H}_{\mu\nu\rho} \equiv \frac{1}{2} \epsilon_{\mu\nu\alpha\beta} H^{\alpha\beta}{}_{\rho}, \quad \tilde{H}_{\mu\nu}{}^{\nu} = 0, \quad (\text{C.5})$$

which are symmetric under $H \leftrightarrow \tilde{H}$. An explicit count of all gauge invariant operators $\partial \dots \partial H$ modulo equations of motion and identities gives⁴¹

$$\mathcal{Z}_{\text{CST},2}(q) = \frac{2q^2(8 - q - q^2)}{(1 - q)^3}, \quad (\text{C.6})$$

which is equivalent to (C.3).

D Partition function of conformal symmetric rank 3 tensor on S^4

The Lagrangian (3.3) for $s = 3$ on unit-radius S^4 background is

$$\mathcal{L}_{\text{CST},3}(S^4) = \nabla^{\lambda} \varphi^{\mu\nu\rho} \nabla_{\lambda} \varphi_{\mu\nu\rho} - \frac{3}{2} \nabla_{\rho} \varphi^{\mu\nu\rho} \nabla^{\lambda} \varphi_{\mu\nu\lambda} + 5 \varphi^{\mu\nu\rho} \varphi_{\mu\nu\rho}. \quad (\text{D.1})$$

Decomposing $\varphi_{\mu\nu\rho}$ as⁴²

$$\begin{aligned} \varphi_{\mu\nu\rho} &= \varphi_{\mu\nu\rho}^{\perp} + \nabla_{(\mu} h_{\nu\rho)}^{\perp} + \nabla_{(\mu} \nabla_{\nu} V_{\rho)}^{\perp} + \nabla_{(\mu} \nabla_{\nu} \nabla_{\rho)} \sigma \\ &\quad - \frac{1}{2} g_{(\mu\nu} V_{\rho)}^{\perp} - \frac{1}{6} g_{(\mu\nu} \nabla^2 V_{\rho)}^{\perp} - g_{(\mu\nu} \nabla_{\rho)} \sigma - \frac{1}{2} g_{(\mu\nu} \nabla_{\rho)} \nabla^2 \sigma, \end{aligned} \quad (\text{D.2})$$

we get

$$\begin{aligned} \mathcal{L}_{\text{CST},3} &= \varphi_{\mu\nu\rho}^{\perp} \Delta_{3\perp}(5) \varphi^{\perp\mu\nu\rho} + \frac{1}{6} h_{\mu\nu}^{\perp} \Delta_{2\perp}(-8) \Delta_{2\perp}(4) h^{\perp\mu\nu} \\ &\quad + \frac{5}{108} V_{\mu}^{\perp} \Delta_{1\perp}(-9) \Delta_{1\perp}(-3) \Delta_{1\perp}(3) V^{\perp\mu}, \end{aligned} \quad (\text{D.3})$$

⁴¹ Note that $\mathcal{Z}_{\text{CST},2} = \mathcal{Z}_2 + \mathcal{Z}_1$ where $\mathcal{Z}_1 = \frac{2q^2(3-q)}{(1-q)^3}$ and $\mathcal{Z}_2 = \frac{2q^2(5-q^2)}{(1-q)^3}$. Here $\mathcal{Z}_s = \frac{2q^2(n_s-q^2)}{(1-q)^3}$ where n_s is the number of physical off-shell d.o.f. (number of components minus gauge parameters).

⁴² Note that the scalar curvature here $R = 12$.

where σ decouples due to scalar gauge invariance. Here $\Delta(M^2) = -\nabla^2 + M^2$ as in (2.10). The Jacobian of transformation (D.2) can be found from

$$\begin{aligned} \int d^4x \sqrt{g} \varphi^{\mu\nu\rho} \varphi_{\mu\nu\rho} &= \int d^4x \sqrt{g} \left[\varphi_{\mu\nu\rho}^\perp \varphi^{\perp\mu\nu\rho} + \frac{1}{3} h_{\mu\nu}^\perp \Delta_{2\perp}(-8) h^{\perp\mu\nu} \right. \\ &\quad \left. + \frac{5}{18} V_\mu^\perp \Delta_{1\perp}(-9) \Delta_{1\perp}(-3) V^{\perp\mu} + \frac{1}{2} \sigma \Delta_0(0) \Delta_0(-4) \Delta_0(-10) \sigma \right]. \end{aligned} \quad (\text{D.4})$$

The resulting partition function is thus (cf. (3.19),(3.20))

$$Z_{\text{CST},3} = \left[\frac{\det \Delta_0(0) \det \Delta_0(-4) \det \Delta_0(-10)}{\det \Delta_{3\perp}(5) \det \Delta_{2\perp}(4) \det \Delta_{1\perp}(3)} \right]^{1/2}. \quad (\text{D.5})$$

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